UNIWERSYTET GDAŃSKI
WYDZIAŁ MATEMATYKI, FIZYKI, I INFORMATYKI

Ray Ganardi

Kierunek : Fizyka
Numer albumu : 271557

Entanglement gain in measurements with unknown outcome

Praca magisterska napisana
pod kierunkiem prof. UG dr. hab. Tomasza Patereka

Gdańsk 2020

Abstract

We characterize the conditions of entanglement increase when a composite quantum system is subjected to a non-selective measurement. In particular, we show that the entanglement of all non-maximally entangled pure states can increase under measurement in a suitable basis. We show Markovian dynamics can implement a measurement asymptotically. Finally, we provide numerical evidence that explains why macroscopic bodies do not spontaneously gain entanglement as a result of decoherence.

Acknowledgements

I would like to express gratitude to my advisor prof. Tomasz Paterek for guidance and never-ending cheerfulness, that makes every discussion enjoyable.

My studies at the University of Gdańsk were supported by a generous stipend from ICTQT, which I kindly acknowledge.

I thank my family, friends, and most importantly my wife, for their continuous support in the past year.

Publication

The thesis is based on the following publication:

Entanglement gain in measurements with unknown results

Margherita Zuppardo, Ray Ganardi, Marek Miller, Somshubhro Bandyopadhyay, and Tomasz Paterek

Phys. Rev. A **99**, 042319 – Published 10 April 2019

Contents

Chapter 1

Introduction

Creating and preserving entanglement between quantum systems is a prerequisite of building a quantum computer [HHHO09]. Typically, this is achieved by minimizing decoherence effects that destroys non-classical correlations [JKP01, SAB+06, BW08, PTSL13, JS17, RWM+18]. Alternatives include engineering the dissipation dynamics such that the steady state is entangled [KMJ+11, VWC09, LGR+13, SHL+13, RRS16].

Here we provide another approach of creating entanglement, through the use of non-selective measurements. In this way, we allow for a larger set of final states compared to the single steady state achievable with dissipation-based approach. The price to pay for this flexibility is the impossibility to create a maximally entangled state.

In the Araki-Żurek model [Zur82, Ara80], system decoherence is modelled by an environment that is continuously measuring the system. Because of wave-function collapse, the measurement projects the system into the pointer basis, thus losing coherence. Now, consider the same picture, but with a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$. Decoherence in the Araki-Żurek model will imply a global measurement on the whole system. This global measurement can increase entanglement within the system.

As a motivating example, suppose we perform a measurement along the Bell basis on two qubits in a state $|\psi\rangle = |00\rangle$. Then with probability $\frac{1}{2}$, we get the outcome $|\phi^+\rangle$, and with probability $\frac{1}{2}$ we get the outcome $|\phi^-\rangle$. We see that in each run of the experiment, we get a maximally entangled state, on average however, we do not produce entanglement because $\rho' = \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{1}{2}|\phi^-\rangle\langle\phi^-|$ is separable.

Instead of performing a Bell measurement, suppose we measure the same system $|\psi\rangle = |00\rangle$ in the following basis

$$\begin{aligned} |\phi_0\rangle &= \tfrac{1}{2}|00\rangle + \tfrac{1}{2}\sqrt{3}\,|11\rangle\,, \\ |\phi_1\rangle &= -\tfrac{1}{2}\sqrt{3}\,|00\rangle + \tfrac{1}{2}|11\rangle\,, \\ |\phi_2\rangle &= |01\rangle\,, \\ |\phi_3\rangle &= |10\rangle\,. \end{aligned} \tag{1.1}$$

The state after the measurement is

$$\rho' = \frac{1}{4}|\phi_0\rangle\langle\phi_0| + \frac{3}{4}|\phi_1\rangle\langle\phi_1|\,,$$

and we gain positive entanglement, e.g. the negativity is $N(\rho') = \frac{\sqrt{3}}{8}$. Since a positive value for negativity indicates non-separability [VW02], we see that the post-measurement state is entangled. Thus we can create entanglement from a pure product state by performing a measurement with unknown outcome.

On one hand, it is not surprising that a global measurement can increase entanglement; it is, after all, a global operation. However generally we expect decoherence to destroy, rather than create correlations. It is thus puzzling if this decoherence can increase entanglement. We will see that if the environment is performing the measurement in a randomly chosen basis, then on average the entanglement will not increase.

The structure of the thesis is as follows: chapter 2 deals with the abstract problem and characterizes the conditions where entanglement can be created. We also give partial results on optimal amounts of entanglement and argue why we expect the bounds to hold in general. Chapter 3 connects the abstract problem with the physical picture. We show Markovian dynamics can implement a projective measurement only in the limit $t \to \infty$. We also provide numerical results showing implausibility of spontaneous gain of entanglement in macroscopic systems..

Notation and terminology

Throughout the thesis, we will assume we are working with a finite-dimensional Hilbert space. We also assume that the dimensions of the Hilbert spaces describing subsystems are equal. Given a basis $\mathcal{B} = \{|\phi_j\rangle\}$, we denote by $P_{\mathcal{B}}$ the rank-one projective measurement in the basis $\mathcal{B}$. Negativity [VW02] is defined as

$$N(\rho) = \frac{1}{2}\left(\|\rho^{T_B}\|_1 - 1\right), \tag{1.2}$$

where $\|A\|$ denotes the trace norm, and A^{T_B} is the partial transpose of the matrix A in some fixed basis.

Chapter 2

Entanglement gain

Consider the following abstract problem. Given an input state ρ, a measurement basis $\mathcal{B}$ and an entanglement measure E, characterize the conditions to obtain

$$E(P_{\mathcal{B}}\rho) > E(\rho).$$

In order to make the calculations tractable, we will use negativity as the entanglement measure.

2.1 Maximally entangled basis

We begin by the following observation. Suppose we quantify entanglement by an entanglement measure E that is convex. Consider a state ρ and a basis $\mathcal{B} = \{|\phi_j\rangle\}$. Because E is convex, we have

$$E(P_{\mathcal{B}}\rho) \leq \sum_j \langle\phi_j|\,\rho\,|\phi_j\rangle\,|\phi_j\rangle\,\langle\phi_j| \leq \max_j E(\phi_j).$$

In other words, the entanglement of the measured state is bounded above by the entanglement of basis states.

If one would like to create the most entanglement, then it is reasonable to expect a projection onto a maximally entangled basis will be the best. This is not the case. Consider the generalized Bell states as

$$|\psi_{jk}\rangle = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\omega_d^{mk}\,|m\rangle\,|m+j\rangle\,,$$

where $j, k = 0, \ldots, d-1,\ \omega_d = e^{i2\pi/d}$, and the sum is taken modulo d. Clearly these states forms a maximally entangled basis. However a projective measurement in this basis can be implemented by LOCC as shown in the theorem below, and thus cannot increase entanglement.

Theorem 1. *Consider two d-dimensional systems. Let P be a measurement in the generalized Bell basis*

$$|\psi_{jk}\rangle = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\omega_d^{mk}\,|m\rangle\,|m+j\rangle\,,$$

where $j,k=0,\dots,d-1,\ \omega_d=e^{i2\pi/d}$. For any input state ρ and any entanglement monotone E, we have

$$E(P\rho)\le E(\rho).$$

Proof. Let us define the operators

$$X=\sum_{l=0}^{d-1}|l+1\rangle\langle l|,$$
$$Z=\sum_{l=0}^{d-1}\omega_d^l|l\rangle\langle l|.$$

Defining $|\Phi\rangle=\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}|mm\rangle$, it is easy to check that

$$|\psi_{jk}\rangle=(\mathbb{1}\otimes X^jZ^k)|\Phi\rangle.$$

Furthermore, we have the relations

$$\mathrm{Tr}\left(X^iZ^j\right)\left(X^kZ^l\right)^\dagger=\delta_{ik}\delta_{jl}d,$$
$$\omega_d^{ij}X^iZ^j=Z^jX^i.$$

Using the relations above, it is tedious but straightforward to check that

$$|\psi_{jk}\rangle\langle\psi_{jk}|=\frac{1}{d^2}\sum_{m,n=0}^{d-1}\omega^{km}\omega^{jn}X^mZ^n\otimes X^mZ^{-n}.$$

Using the identity $\sum_{i=0}^{d-1}\omega_d^{i(j-k)}=d\delta_{jk}$, we find that

$$\begin{aligned}P\rho&=\sum_{j,k=0}^{d-1}|\psi_{jk}\rangle\langle\psi_{jk}|\rho|\psi_{jk}\rangle\langle\psi_{jk}|\\&=\sum_{j,k=0}^{d-1}\sum_{\substack{m,n=0\\m',n'=0}}^{d-1}\frac{1}{d^4}\omega^{k(m-m')}\omega^{j(n-n')}(X^mZ^n\otimes X^mZ^{-n})\rho(X^{m'}Z^{n'}\otimes X^{m'}Z^{-n'})^\dagger\\&=\sum_{\substack{m,n=0\\m',n'=0}}^{d-1}\frac{1}{d^2}\delta_{mm'}\delta_{nn'}(X^mZ^n\otimes X^mZ^{-n})\rho(X^{m'}Z^{n'}\otimes X^{m'}Z^{-n'})^\dagger\\&=\sum_{m,n=0}^{d-1}\frac{1}{d^2}(X^mZ^n\otimes X^mZ^{-n})\rho(X^mZ^n\otimes X^mZ^{-n})^\dagger.\end{aligned}$$

Clearly, the map has an LOCC implementation. Alice generates a pair of uniformly random integers m,n that ranges from 0 to $d-1$, and communicates the result to Bob. Alice then applies the unitary X^mZ^n to her system, while Bob applies X^mZ^{-n}. Then both Alice and Bob erase the record of the values of m,n. Therefore for any entanglement monotone E, we have $E(P\rho)\le E(\rho)$. □

The next lemma strengthens the result by showing that any maximally entangled basis on two qubits is equivalent under local unitaries to the Bell basis.

Lemma 2. *Let $\mathcal{B} = \{|\psi_j\rangle\}$ be an orthonormal basis on two qubits, with all elements maximally entangled. Then the basis is equivalent to the Bell basis under some local unitary.*

Proof. Let $\{|a_k b_l\rangle\}$ be the Schmidt basis of $|\psi_0\rangle$. We expand the elements $|\psi_j\rangle$ as follows

$$|\psi_j\rangle = \sum_{k,l=0}^{1} \alpha_{kl}^{(j)} |a_k b_l\rangle ,$$

and consider 2×2 complex matrices $\alpha^{(j)}$ with entries defined above. Recall that the Schmidt coefficients of $|\psi_j\rangle$ are simply the singular values of the matrix $\alpha^{(j)}$ [NC09]. Since $|\psi_j\rangle$ are maximally entangled, the singular values of $\alpha^{(j)}$ are all equal to $\frac{1}{\sqrt{2}}$, and it follows that $\alpha^{(j)}$ is proportional to a unitary matrix. Note that they are also orthogonal in the Hilbert-Schmidt inner product.

We can write a traceless 2×2 unitary matrix as a Hermitian matrix multiplied by a phase factor. Since $\operatorname{Tr} \alpha^{(j)} = 0$ for $j = 1, 2, 3$, we can find $A^{(j)}$ Hermitian and η_j real such that $\alpha^{(j)} = e^{i\eta_j} A^{(j)}$. Multiplying the matrices $\alpha^{(j)}$ by a phase $e^{-i\eta_j}$ does not change the projection basis $|\psi_j\rangle \langle\psi_j|$, so without loss of generality we can assume that $\alpha^{(j)}, j = 1, 2, 3$ is Hermitian.

Identifying $\alpha^{(0)} = \frac{1}{\sqrt{2}}\mathbb{1}$, we have an orthonormal basis $\{\alpha^{(j)} \mid j = 0, 1, 2, 3\}$ of the four-dimensional real Hilbert space of Hermitian 2×2 matrices with the Hilbert-Schmidt inner product. The same space is spanned by the identity and the Pauli matrices, and therefore there exists an isometry R such that

$$\alpha^{(j)} = R\left(\frac{1}{\sqrt{2}}\sigma_j\right),$$

where $\sigma_0 = \mathbb{1}$, and σ_j is the j-th Pauli matrix. The matrix representation of the isometry R is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & R_0 \end{pmatrix},$$

where $R_0 \in O(3)$ is an orthogonal matrix.

We will assume that $R_0 \in SO(3)$. This can be done without loss of generality, because if $\det R_0 = -1$, we can take one of the basis vectors, say $|\psi_1\rangle$ with an opposite sign. This amounts to a global phase change, which does not change the projector $|\psi_1\rangle \langle\psi_1|$. It does however change the sign of $\alpha^{(1)}$ and consequently the sign of $\det R_0$.

Using the homomorphism between the groups $SU(2)$ and $SO(3)$, we infer that there exists a unitary matrix $U \in SU(2)$ such that $\alpha^{(j)} = \frac{1}{\sqrt{2}} R_0 \sigma_j = \frac{1}{\sqrt{2}} U\sigma_j U^\dagger$ for $j = 1, 2, 3$, and therefore

$$|\psi_j\rangle = \frac{1}{\sqrt{2}} \sum_{k,l=0}^{1} \left(U\sigma_j U^\dagger\right)_{kl} |a_k b_l\rangle .$$

Note that $\alpha^{(0)} = \mathbb{1}$, and therefore the last equation holds for all j. Define the new local basis

$$|m\rangle = \sum_k U_{km} |a_k\rangle ,$$

$$|n\rangle = \sum_l U_{ln}^* |b_l\rangle .$$

We can thus write $|\psi_j\rangle$ as

$$|\psi_j\rangle = \frac{1}{\sqrt{2}} \sum_{m,n=0}^{1} (\sigma_j)_{mn} |mn\rangle ,$$

which is the standard Bell basis. □

Thus we conclude that a projection on a maximally entangled basis in a two qubit system cannot increase entanglement. In higher dimensions, we could not show that any maximally entangled basis is can be transformed into the generalized Bell basis by local unitaries. So our result only shows the existence of a maximally entangled basis that cannot increase entanglement.

2.2 Conditions for entanglement gain

We show that it is almost always possible to increase the negativity of a pure state.

Theorem 3. *Let $|\phi\rangle$ be a non-maximally entangled pure state. There exists a basis $\mathcal{B}$ such that a measurement in $\mathcal{B}$ increases negativity, i.e.*

$$N(P_{\mathcal{B}}\phi) > N(\phi)$$

Proof. Let $|\Phi\rangle$ be the maximally entangled state with the same Schmidt basis as $|\phi\rangle$. We define

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2 + 2\langle\phi|\Phi\rangle}} \left(|\phi\rangle + |\Phi\rangle\right), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2 - 2\langle\phi|\Phi\rangle}} \left(|\phi\rangle - |\Phi\rangle\right), \end{aligned}$$

and we note that they are orthonormal. Thus we pick them as the first two vectors in our measurement basis. Furthermore, since $|\phi\rangle$ can be expressed as a linear combination of $|\psi_0\rangle$ and $|\psi_1\rangle$, the rest of the basis can be chosen arbitrarily, since they must be orthogonal to $|\phi\rangle$. Let this basis be $\mathcal{B}$.

It is easy to verify that the state after the measurement is given by

$$\begin{aligned} P_{\mathcal{B}}\phi &= |\langle\phi|\psi_0\rangle|^2 |\psi_0\rangle\langle\psi_0| + |\langle\phi|\psi_1\rangle|^2 |\psi_1\rangle\langle\psi_1| \\ &= \tfrac{1}{2}|\phi\rangle\langle\phi| + \tfrac{1}{2}|\Phi\rangle\langle\Phi| . \end{aligned}$$

Since $|\phi\rangle$ and $|\Phi\rangle$ has the same Schmidt basis, it follows that

$$\begin{aligned} N(P_{\mathcal{B}}\phi) &= N\left(\tfrac{1}{2}|\phi\rangle\langle\phi| + \tfrac{1}{2}|\Phi\rangle\langle\Phi|\right) \\ &= \tfrac{1}{2}N(\phi) + \tfrac{1}{2}N(\Phi) \end{aligned}$$

which must be larger than $N(\phi)$. □

With this result, we might ask the converse question: given a measurement basis, is there always a state whose entanglement will increase? The answer is of course no. If the measurement basis consists of solely product vectors, then the output state must be separable, and the final entanglement will be zero. Another counterexample was shown in Theorem 1, where a measurement in a maximally entangled basis cannot increase entanglement.

It turns out that all these examples are measure zero, and we can find a pure state whose negativity will increase for almost all bases.

Theorem 4. *Let $\mathcal{B} = \{|\psi_j\rangle\}$ be a basis, with one element $|\psi_0\rangle$ having two non-zero Schmidt coefficients that are not equal. There exists a pure state $|\phi\rangle$ such that a measurement in $\mathcal{B}$ increases negativity, i.e.*

$$N\left(P_{\mathcal{B}}\phi\right) > N(\phi)$$

Proof. Let d be the number of non-zero Schmidt coefficients of $|\psi_0\rangle$. By assumption, $|\psi_0\rangle = \sum_{i=0}^{d-1} \sqrt{p_i}\,|ii\rangle$, with $p_i > 0$ and $p_0 \neq p_1$. We will construct the input state by rotating the vector $|\psi_0\rangle$. We choose the input state as follows

$$|\phi\rangle \propto |\psi_0\rangle - \varepsilon\,|\Phi\rangle = \sum_{i=0}^{d-1} \left(\sqrt{p_i} - \frac{\varepsilon}{\sqrt{d}}\right) |ii\rangle\,, \tag{2.1}$$

where $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ is defined using the Schmidt basis of $|\psi_0\rangle$. We will show that there is a range of $\varepsilon > 0$ such that $N(P_{\mathcal{B}}\phi) > N(\phi)$.

Suppose we choose ε small such that

$$0 < \frac{\varepsilon}{\sqrt{d}} < \min_i p_i, \tag{2.2}$$

so all the coefficients in eq. (2.1) are positive. With some effort, we find the negativity of the initial state $|\phi\rangle$ is given by

$$N(\phi) = \frac{1}{C^2}\left[N(\psi_0) + \varepsilon(\varepsilon - 2\beta)N(\Phi)\right],$$

where $\beta = \langle\psi_0|\Phi\rangle$, and $C = (1 + \varepsilon^2 - 2\varepsilon\beta)^{1/2}$.

The post-measurement state is

$$P_{\mathcal{B}}\phi = (1-p)\,|\psi_0\rangle\,\langle\psi_0| + p\sigma,$$

where σ is orthogonal to $|\psi_0\rangle\,\langle\psi_0|$, and $1 - p = |\,\langle\psi_0|\phi\rangle\,|^2 = \left(\frac{1-\varepsilon\beta}{C}\right)^2$. By triangle inequality on trace norm, we have

$$\begin{aligned} \||\psi_0\rangle\,\langle\psi_0|^{T_B}\|_1 &= \|(P_{\mathcal{B}}\phi + p(|\psi_0\rangle\,\langle\psi_0| - \sigma))^{T_B}\|_1 \\ &\leq \|(P_{\mathcal{B}}\phi)^{T_B}\|_1 + p\|\psi_0{}^{T_B} - \sigma^{T_B}\|_1 \\ &\leq \|(P_{\mathcal{B}}\phi)^{T_B}\|_1 + 2dp. \end{aligned}$$

Recall that negativity is given by $N(\rho) = \frac{1}{2}(\|\rho^{T_B}\|_1 - 1)$. Thus we find that

$$\begin{aligned} N(P_{\mathcal{B}}\phi) &\geq N(\psi_0) - pd \\ &= N(\psi_0) - d\left(1 - \left(\frac{1-\varepsilon\beta}{C}\right)^2\right) \\ &= N(\psi_0) - d\left(\frac{\varepsilon^2(1-\beta^2)}{C^2}\right). \end{aligned}$$

We wish to choose ε such that $N(P_{\mathcal{B}}\phi) > N(\phi)$. By the lower bound above, it is enough to choose ε such that

$$N(\psi_0) - d\left(\frac{\varepsilon^2(1-\beta^2)}{C^2}\right) > \frac{1}{C^2}\left[N(\psi_0) + \varepsilon(\varepsilon - 2\beta)N(\Phi)\right].$$

Multiplying by C^2, and substituting $C^2 = (1 + \varepsilon^2 - 2\varepsilon\beta)$, we get the condition

$$\begin{aligned} &(1+\varepsilon^2 - 2\varepsilon\beta)N(\psi_0) - d\varepsilon^2(1-\beta^2) > N(\psi_0) + \varepsilon(\varepsilon - 2\beta)N(\Phi) \\ \iff\ &\varepsilon\left(N(\Phi) - N(\psi_0) + d(1-\beta^2)\right) < 2\beta(N(\Phi) - N(\psi_0)). \end{aligned} \tag{2.3}$$

By assumption, the first two Schmidt coefficients of $|\psi_0\rangle$ are not equal, therefore it cannot be a maximally entangled state in its subspace. Therefore we have $N(\Phi) > N(\psi_0)$.

Clearly, there exists a choice of ε satisfying eq. (2.2) and (2.3), and for such an ε, we have $N(P_{\mathcal{B}}\phi) > N(\phi)$. □

The conditions in the theorem above might seem arbitrary, but it is in fact necessary. Consider the following basis

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \\ |\psi_4\rangle &= |02\rangle, \\ |\psi_5\rangle &= |12\rangle, \\ |\psi_6\rangle &= |20\rangle, \\ |\psi_7\rangle &= |21\rangle, \\ |\psi_8\rangle &= |22\rangle. \end{aligned}$$

A measurement in this basis has an LOCC implementation as follows, and therefore it cannot increase entanglement. Let $\Pi_{01} = |0\rangle\langle 0| + |1\rangle\langle 1|$, $\Pi_i = |i\rangle\langle i|$. Alice performs a measurement in $\{\Pi_{01}, \Pi_2\}$ and notes the outcome. If Alice obtained the outcome Π_2, then Bob performs a measurement in $\{\Pi_0, \Pi_1, \Pi_2\}$, and they both prepare the state $\{|\psi_6\rangle, |\psi_7\rangle, |\psi_8\rangle\}$, depending on Bob's measurement result. If Alice obtained the outcome Π_{01}, then Bob performs a measurement in $\{\Pi_{01}, \Pi_2\}$ and notes the outcome. If Bob obtained the outcome Π_{01}, then they should perform the LOCC protocol to implement non-selective measurement in a Bell basis (as in Theorem 1). Otherwise, Bob obtained the outcome Π_2, and Alice should perform another measurement in $\{\Pi_0, \Pi_1\}$, and they both prepare the state $\{|\psi_4\rangle, |\psi_5\rangle\}$.

2.3 Conditions for impossibility of entanglement gain

We could ask if similar results hold if we allow the input states to be mixed states. The answer is clearly no. To see this, note that a maximally mixed state is invariant under a measurement, thus its entanglement cannot be increased in this fashion.

More generally, recall that we say ρ is *absolutely separable* if $U\rho U^\dagger$ is separable for any unitary U acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$. We show that any rank-one projective measurement can be implemented as a mixed unitary channel.

Theorem 5. *For any rank-one projective measurement $\rho \mapsto \sum_j \Pi_j \rho \Pi_j$, there exists a collection of unitaries U_j and weights p_j such that*

$$\sum_{j=0}^{d-1} \Pi_j \rho \Pi_j = \sum_{j=0}^{d-1} p_j U_j \rho U_j^\dagger$$

Proof. By construction. Let us define

$$U = \sum_{j=0}^{d-1} \omega_d^j \Pi_j,$$

where $\omega_d = e^{i2\pi/d}$. We have

$$\begin{aligned} \frac{1}{d}\sum_{i=0}^{d-1} (U^i)\rho(U^i)^\dagger &= \frac{1}{d}\sum_{i,j,k=0}^{d-1} \omega_d^{i(j-k)} \Pi_j \rho \Pi_k \\ &= \sum_{j,k=0}^{d-1} \delta_{jk} \Pi_j \rho \Pi_k \\ &= \sum_{k=0}^{d-1} \Pi_k \rho \Pi_k, \end{aligned}$$

where we used the identity $\sum_{i=0}^{d-1} \omega_d^{i(j-k)} = d\delta_{jk}$ □

Thus the entanglement of any absolutely separable state cannot be increased with a non-selective measurement.

Note that the following proof does not work. A rank-one projective measurement is a unital, trace-preserving map. Extreme points of bistochastic map are given by unitaries, and therefore any bistochastic map can be expressed as a convex combination of unitaries. The reason is there exist unital channels that are not a convex combination of unitaries [SVW05]. The classical analogue of this is solved by the well-known Birkhoff's theorem, which states that any doubly stochastic matrix is a convex combination of permutations. The quantum problem turns out to be harder, and solved in the negative in [HM11].

Finally, let us show that we can increase the entanglement of states that are ε-close to the set of absolutely separable states. The set of absolutely separable states has been characterized for two qubits[VAM01, IH00]. Consider a two qubit state ρ. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are eigenvalues of ρ, then ρ is absolutely separable if and only if $\lambda_3 + 2\sqrt{\lambda_2\lambda_4} - \lambda_1 \geq 0$. Consider $\rho = \frac{1+2\varepsilon}{3} |00\rangle \langle 00| + \frac{1-\varepsilon}{3} |11\rangle \langle 11| + \frac{1-\varepsilon}{3} |01\rangle \langle 01|$. The state ρ is obviously separable, and ε-close to the set of absolutely separable states. Performing a measurement in the basis defined in eq. (1.1), we see that the negativity of the post-measurement state is always strictly positive.

2.4 Optimality

An interesting question is how much can we increase the entanglement of a given pure state? Suppose we found an initial state ϕ and a measurement basis $\mathcal{B}$ such that $P_{\mathcal{B}}\phi$ is maximally entangled. A maximally entangled state must be a pure state [HHHO09]. If $P_{\mathcal{B}}\phi$ is pure, then we must have $\phi \in \mathcal{B}$. Therefore $P_{\mathcal{B}}\phi = \phi$, and the state ϕ itself must be maximally entangled.

On the other hand, Theorem 2.2 shows that we can increase the negativity of a pure product state to half of its maximum value. Characterizing the basis giving the largest increase is a complex problem, but we can characterize the largest increase on a certain family of basis. In the following section, we show that for this family of basis, a halfway-to-maximum increase is the best that we can do. We also provide numerical evidence suggesting that this holds for all bases.

Consider an arbitrary pure state $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with its Schmidt decomposition

$$|\phi\rangle = \sum_{i=0}^{d-1} \sqrt{p_i} |ii\rangle .$$

Let $\mathcal{W}$ be the d-dimensional subspace that is spanned by the Schmidt basis of $|\phi\rangle$. We can decompose $\mathcal{H}_A \otimes \mathcal{H}_B$ into a direct sum $\mathcal{W} \oplus \mathcal{W}_\perp$, where $\mathcal{W}_\perp$ is the subspace orthogonal to $\mathcal{W}$.

Consider the family of bases for $\mathcal{H}_A \otimes \mathcal{H}_B$ that is a concatenation of a basis for $\mathcal{W}$ and a basis for $\mathcal{W}_\perp$. Let $\mathcal{B}$ be an arbitrary basis from this family. The basis vectors for $\mathcal{W}$ can be written as

$$|\psi_j\rangle = \sum_{i=0}^{d-1} \alpha_{ji} |ii\rangle . \tag{2.4}$$

Note that $|\phi\rangle \in \mathcal{W}$, so the choice of basis for $\mathcal{W}_\perp$ will not affect the post-measurement state. Performing a measurement on $|\phi\rangle$, the post-measurement state is

$$\begin{aligned} P_{\mathcal{B}}\phi &= \sum_{j=0}^{d-1} |\langle \phi|\psi_j\rangle|^2 |\psi_j\rangle \langle \psi_j| \\ &= \sum_{j,i,i'} \sum_{k,k'} \sqrt{p_i}\alpha_{ji}\sqrt{p_{i'}}\alpha^*_{ji'}\alpha_{jk}\alpha^*_{jk'} |kk\rangle \langle k'k'| . \end{aligned}$$

Note that the post-measurement state $P_{\mathcal{B}}\phi$ is of the form $\sigma = \sum_{ij} a_{ij} |ii\rangle \langle jj|$, i.e. maximally correlated [Rai99, VSPM01]. For a maximally correlated state σ, negativity is given by the simple formula [KKS07]

$$N(\sigma) = \sum_{i<j} |a_{ij}|.$$

Thus we can bound the negativity of $P_{\mathcal{B}}\phi$ as

$$\begin{aligned} N(P_{\mathcal{B}}\phi) &= \sum_{k<k'} \left| \sum_{j,i,i'} \sqrt{p_i}\alpha_{ji}\sqrt{p_{i'}}\alpha^*_{ji'}\alpha_{jk}\alpha^*_{jk'} \right| \\ &\geq \left| \sum_{k<k'} \sum_{j,i,i'} \sqrt{p_i}\alpha_{ji}\sqrt{p_{i'}}\alpha^*_{ji'}\alpha_{jk}\alpha^*_{jk'} \right|. \end{aligned} \tag{2.5}$$

Define the following matrices

$$\begin{aligned} (\mathbf{m})_{ii'} &= \sum_{k<k'} \sum_{j} \alpha_{ji}\alpha^*_{ji'}\alpha_{jk}\alpha^*_{jk'}, \\ (\mathbf{q})_i &= \sqrt{p_i}, \end{aligned}$$

where $\mathbf{m}$ characterizes the measurement basis, and $\mathbf{q}$ characterizes the input state. We can write eq. (2.5) simply as

$$N(P_{\mathcal{B}}\phi) \geq |\langle \mathbf{q}, \mathbf{mq} \rangle|.$$

As an example, let us study this representation for two qubits and what we can say about an upper bound on entanglement gain. Let $|\phi\rangle$ be a pure two-qubit state

$$|\phi\rangle = a\,|00\rangle + \sqrt{1-a^2}\,|11\rangle\,,$$

where $a \in \left[0, \frac{1}{\sqrt{2}}\right]$ without loss of generality. We choose the measurement basis

$$|\psi_0\rangle = b\,|00\rangle + \sqrt{1-b^2}\,|11\rangle\,, \tag{2.6}$$

$$|\psi_1\rangle = \sqrt{1-b^2}\,|00\rangle - b\,|11\rangle\,, \tag{2.7}$$

where $b \in [0, 1]$. The other basis elements can be chosen arbitrarily since they are orthogonal to $|\phi\rangle$. Using the previously introduced notation, we have

$$\begin{aligned} \mathbf{q} &= \begin{pmatrix} a \\ \sqrt{1-a^2} \end{pmatrix}, \\ \mathbf{m} &= \begin{pmatrix} b\sqrt{1-b^2}(2b^2-1) & 2b^2(1-b^2) \\ 2b^2(1-b^2) & -b\sqrt{1-b^2}(2b^2-1) \end{pmatrix}. \end{aligned}$$

Both $|\psi_0\rangle$ and $|\psi_1\rangle$ are characterized by a single parameter b, and they have the same Schmidt coefficient. Therefore, they must have the same negativity. Let us denote $N_b = b\sqrt{1-b^2}$ the negativity of the basis. Notice that for two qubits, the outer sum in eq. (2.5) only has a single term,

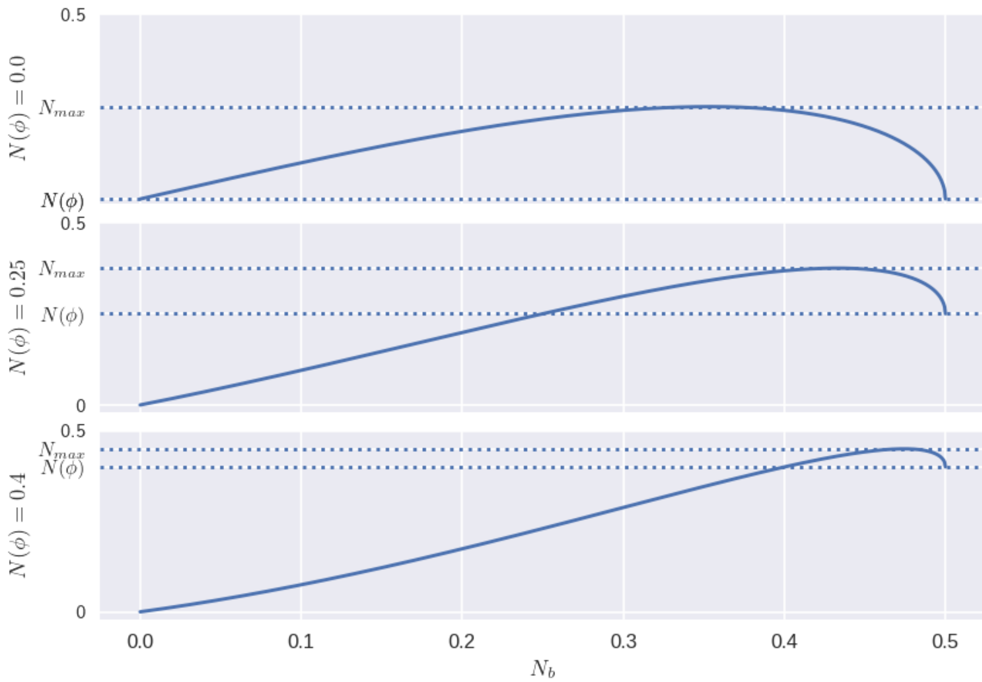

Figure 2.1: The negativity of the post-measurement state $N(P_{\mathcal{B}}\phi)$ computed from eq. (2.8).

and thus the inequality becomes an equality. The negativity of the post-measurement state $P_{\mathcal{B}}\phi$, expressed in terms of $N(\phi)$ and N_b is

$$N(P_{\mathcal{B}}\phi) = N_b\sqrt{1-4N(\phi)^2}\sqrt{1-4N_b^2} + 4N(\phi)N_b^2. \tag{2.8}$$

Differentiating eq. (2.8) w.r.t N_b and setting it to zero, we obtain the maximum value of $N(P_{\mathcal{B}}\phi)$

$$N_{\max} = \tfrac{1}{2}\left(N(\phi) + \tfrac{1}{2}\right). \tag{2.9}$$

Noting that the maximum negativity of a two qubit state is $\frac{1}{2}$, we find that the gain in negativity is at most $\frac{1}{2}\left(\frac{1}{2} - N(\phi)\right)$.

This implies that the input state that gain the most negativity is a product state $|\phi\rangle = |00\rangle$ (see Figure 2.1). The optimal measurement basis $\mathcal{B}$ is

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{4+2\sqrt{2}}}\left(\left(\sqrt{2}+1\right)|00\rangle + |11\rangle\right), \\
|\psi_1\rangle &= \frac{1}{\sqrt{4-2\sqrt{2}}}\left(\left(\sqrt{2}-1\right)|00\rangle - |11\rangle\right), \\
|\psi_1\rangle &= |01\rangle, \\
|\psi_3\rangle &= |10\rangle,
\end{aligned}$$

with negativity increasing from $N(\phi) = 0$ to $N(P_{\mathcal{B}}\phi) = \frac{1}{4}$.

Of course, we have only shown that this is the optimum gain in negativity for this family of measurement bases. We could not obtain a proof that this is the optimal choice of basis, even for two qubits. However, a numerical study supports our conclusion. Fig. 2.2 presents the results of one million samples of pure state ϕ and measurement basis $\mathcal{B}$, sampled independently and uniformly from the Haar measure.

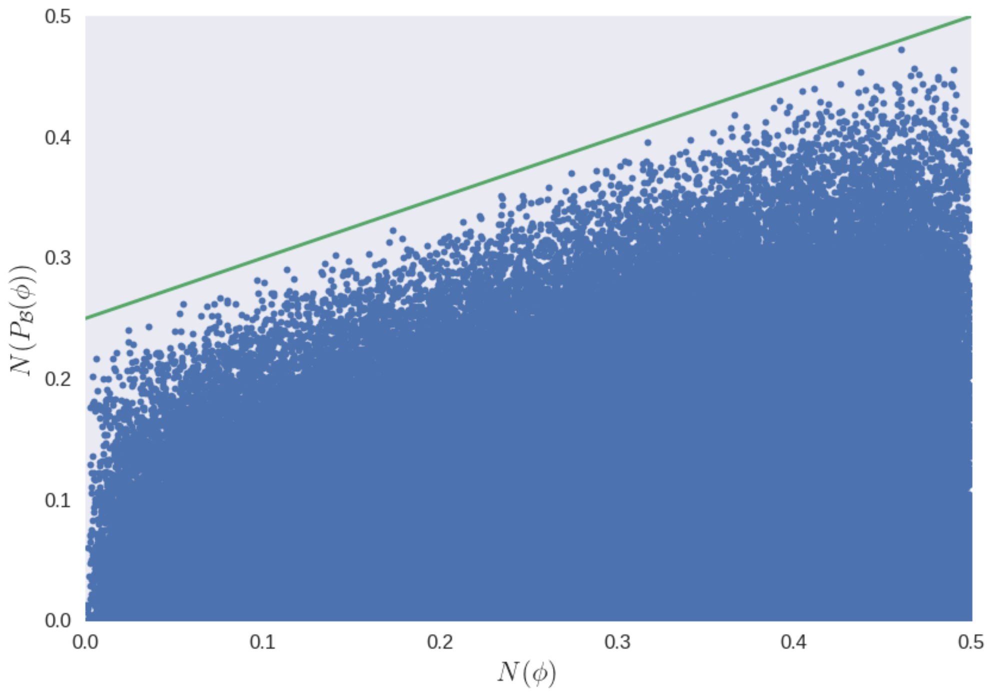

Figure 2.2: Negativity of ϕ and $P_{\mathcal{B}}\phi$. Each point corresponds to a state ϕ and a measurement basis $\mathcal{B}$ sampled independently and uniformly according to the Haar measure. The upper boundary is given by eq. (2.9).

Chapter 3

Decoherence

We have shown that a global, non-selective measurement can increase entanglement in a bipartite system. In this chapter, we ask if such a map can be realized in Markovian dynamics. We also argue why an entanglement gain is unlikely to be observed in macroscopic systems.

3.1 Markovian dynamics

As a simple example, note that the equation

$$\rho_t = e^{-\lambda t}\rho + \left(1 - e^{-\lambda t}\right)\sum_j \Pi_j \rho \Pi_j,$$

with $\lambda > 0$ defines a Markovian evolution. To see this, we observe that

$$\frac{d\rho_t}{dt} = -\lambda \left(\rho - \sum_j \Pi_j \rho \Pi_j\right),$$

which has the Lindblad form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_{j=0}^{d^2-1}\left(L_j \rho L_j^\dagger - \tfrac{1}{2}\{L_j^\dagger L_j, \rho\}\right)$$

with no Hamiltonian and Lindblad operators $L_j = \sqrt{\lambda}\Pi_j$. It is easy to see that asymptotically, the evolution approaches a projection $\lim_{t\to\infty}\rho_t = P_{\{\Pi_j\}}\rho$. Therefore it is possible to have a Markovian dynamics that approaches a projective map asymptotically.

To illustrate how a more realistic dynamics can lead to a projective measurement, let us take concrete example within the Araki-Żurek model [Ara80, Zur82]. A Markovian map $\mathcal{E}_t : B(\mathcal{H}) \to B(\mathcal{H})$ describes the evolution of an open system. Recall that Markovianity means $\mathcal{E}_{s+t} = \mathcal{E}_s \circ \mathcal{E}_t$. We also require that $\mathcal{E}_0 = \mathbb{1}$, and the map to be norm-continuous. There exists a Markovian map that implements a non-selective measurement in the limit [BO03]. With a simple modification, there also exists a Markovian map that implements a non-selective measurement in the limit, and increases the negativity of some state.

Consider a system of two qubits, coupled to an environment represented by a free particle moving on a line. Suppose initially the system is in a product state $\rho = \rho_S \otimes \omega_E$, where $\rho_S = |00\rangle \langle 00|$ and the initial state of the environment is $\omega_E = |\phi_E\rangle \langle \phi_E|$, where

$$\phi_E(x) = \frac{1}{\sqrt{2\pi}} \int \frac{e^{ipx}}{\sqrt{\pi(1+p^2)}}\, dp.$$

ϕ_E is chosen as a Cauchy distribution in momentum space to ensure that the evolution is Markovian [BO03]. We choose the Hamiltonian of the form

$$H_{SE} = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + A \otimes B,$$

where $H_S = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z$ is the Hamiltonian of the system, $H_E = \hat{p}^2$ is the Hamiltonian of the environment, and $\hat{p}$ is the momentum operator of the environment. We encode the choice of the measurement basis in the interaction term $A \otimes B$. We would like the reduced dynamics on the system to go to a projection onto the basis defined in eq. (2.7). We choose $B = \hat{p}$. The operator A determines the projectors, therefore, we choose $A = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|$ with non-degenerate λ_j to make all the projectors rank one.

This choice of initial state and dynamics implies the evolution of the system is given by (eq. (7) in [BO03])

$$\rho_t = \sum_{m,n=0}^{3} e^{-|\lambda_m - \lambda_n| t} e^{i(\gamma_n - \gamma_m)t} \langle \psi_n| \rho_0 |\psi_m\rangle |\psi_n\rangle \langle \psi_m| ,$$

where $\gamma_0 = \gamma_1 = 0$ and $\gamma_2 = -\gamma_3 = 2$. Simplifying, we get

$$\rho_t = b^2 |\psi_0\rangle \langle \psi_0| + (1-b^2) |\psi_1\rangle \langle \psi_1| + e^{-|\mu| t} b\sqrt{1-b^2} \left(|\psi_0\rangle \langle \psi_1| + |\psi_1\rangle \langle \psi_0|\right),$$

where $\mu = \lambda_1 - \lambda_2 \neq 0$. The negativity of ρ_t is

$$N(\rho_t) = b\sqrt{1-b^2} |2b^2 - 1| \left(1 - e^{-|\mu| t}\right).$$

It is easy to see that $\rho_t \to P_{\mathcal{B}}\rho$ in the limit $t \to \infty$ in trace norm. Note also that the convergence is exponentially fast, scaling as $e^{-|\mu| t}$.

It might be interesting to have a Markovian dynamics that implements a non-selective measurement at a finite time. However, such an evolution cannot exists, at least on a finite-dimensional system.

Theorem 6. *Let Π be a projection map with rank one projectors $\{\Pi_j\}$. There are no $\mathcal{E}_t : M_n \to M_n$ Markovian map, with $\mathcal{E}_0 = \mathbb{1}$, such that $\mathcal{E}_\tau = \Pi$ for some $\tau < \infty$.*

Proof. By contradiction. Suppose there is a finite τ such that $\mathcal{E}_\tau = \Pi$. Let $\Pi' = \mathbb{1} - \Pi$, where $\mathbb{1}$ is the identity map. Let $\mathcal{E}'_t = \Pi' \circ \mathcal{E}_t$, and note that $\mathcal{E}'_\tau = 0$.

We will show that $\mathcal{E}'_t = 0$ for all $t < \tau$, thus $\mathcal{E}_t = \mathcal{E}'_t + \Pi \circ \mathcal{E}_t$ is a projector for all $t < \tau$. The Markovianity of $\mathcal{E}_t$ and $\mathcal{E}_\tau = \Pi$ implies $\mathcal{E}_t \circ \Pi = \Pi \circ \mathcal{E}_t$, which in turn implies that $\mathcal{E}'_t$ is also Markovian. Since $\mathcal{E}'_t$ is a linear transformation, it can be represented as an $n^2 \times n^2$ matrix. Let us fix an integer m and study the matrix representation A of the map $\mathcal{E}'_{\tau/m}$. Because $\mathcal{E}'_t$ is Markovian and $\mathcal{E}'_\tau = 0$, we conclude that $A^m = 0$. This implies all the eigenvalues of A are zero and it has a simple Jordan normal form

$$A = S \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} S^{-1},$$

with S an invertible matrix. Since matrix A is a $n^2 \times n^2$ matrix, there exists an integer p for which $A^p = 0$, because multiplying A by itself shifts the 1s further from the diagonal. We therefore have $\mathcal{E}'_{\frac{p}{m}\tau} = 0$. Since $p < n^2$ independent of m, we have $\mathcal{E}_{q\tau}$ for any rational number q. Using the Markovianity of $\mathcal{E}'_t$, we can extend this to all strictly positive real numbers. Therefore we have shown that $\mathcal{E}_t = \Pi$ for all $0 < t < \tau$. Since the map $t \to \mathcal{E}_t$ is continuous, this implies $\mathcal{E}_0 = \Pi$, which contradicts the assumption $\mathcal{E}_0 = \mathbb{1}$. □

3.2 Randomness and spontaneous gain

When we prepare a composite system in a product state $\rho = \rho_A \otimes \rho_B$ and let it decohere, generally we do not observe spontaneous entanglement gain. However, theorem 4 suggests that for a random interaction, we must have some entanglement gain for some pure state. Note that the set of measurement basis can be identified with the set of unitaries. The set of unitaries can in turn be given a measure invariant under multiplication, turning it into a probability space. Theorem 4 shows that the set of basis that exhibits entanglement gain is full measure. Therefore if we pick a random measurement basis, we will be able to find a pure state whose entanglement will increase after measurement.

We explain this by performing a numerical study with a random pure initial states and a random measurement basis. We found numerically that the probability of negativity gain in this scenario is very small for a system of two qubits ($\approx 1.6\%$). For higher dimensional systems, the probability is even smaller ($< 10^{-6}$).

Roughly, as the dimension of the system increases, the average negativity of a random pure state increases, whereas the average negativity of the post-measurement state increases at a much slower rate (see Figure 3.1). The increase of average negativity of a random pure state has been noted in [Dat10], where it was shown that asymptotically, the average negativity of a random pure state goes to a constant fraction of the maximum negativity. First, we notice that the distribution of negativity in the post-measurement state is almost independent of the input state. We estimate the independence as follows. We generate pure states $|\phi\rangle$ uniformly from the Haar measure. Then for each pure state, we generate measurement basis $\mathcal{B}$ independently and uniformly from the Haar measure.

Let $\mu_{d\times d}$ be the Haar measure over the measurement basis, and $\nu_{d\times d}$ the Haar measure over pure states of dimension $d \times d$. Let $f(x, \phi)$ be the pdf of negativity of the post-measurement state, given the input state is ϕ

$$f(x, \phi) = \int \delta(N(P_{\mathcal{B}}\phi) - x)\ d\mu_{d\times d}(\mathcal{B}).$$

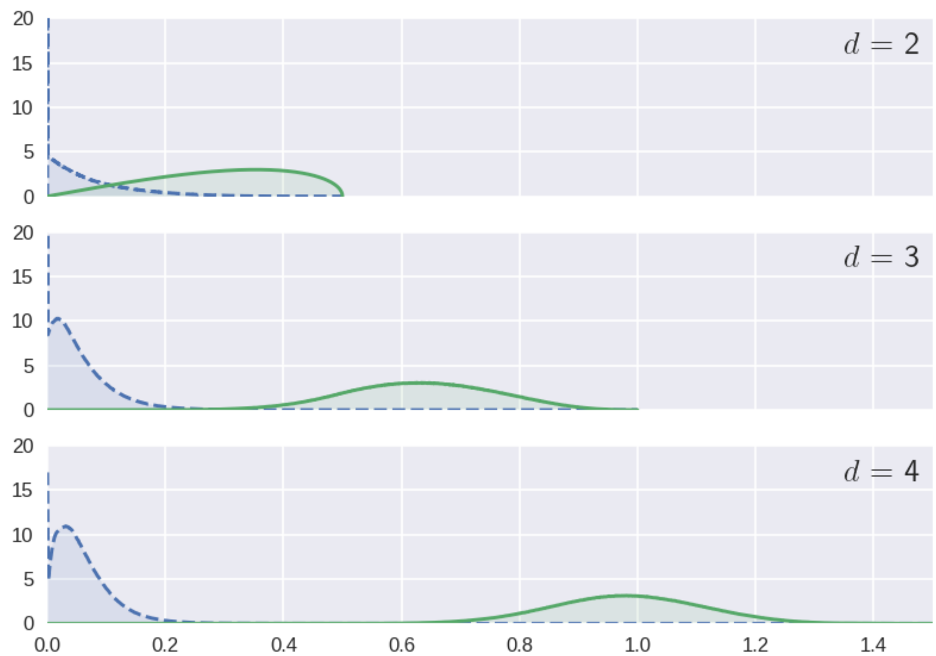

Figure 3.1: Distribution of negativity between two d-level systems. The distributions of post-measurement negativity are shown in blue, whereas the distributions of negativity of random pure states are shown in green. The distribution of negativity of random pure states are estimated by sampling 10^7 states. The distribution of post-measurement negativity is estimated by sampling 10^4 measurement basis and 10^3 pure input states.

d	$\mathcal{S}_d$
2	0.063
3	0.054
4	0.045

Table 3.1: Average statistical distance eq. (3.1) obtained by sampling 10^3 input states, 10^4 measurement basis per state. It shows that the distribution of negativity in the post-measurement state is almost independent of the input state.

Let $g(x)$ be the pdf of negativity when we average over all input states ϕ

$$g(x) = \int f(x, \phi) \; d\nu_{d\times d}(\phi).$$

We can quantify the independence of f on ϕ by averaging the total variation/L^1 distance [NC09, Tsy09] between the two

$$\mathcal{S}_d = \int \int \tfrac{1}{2} \left| f(x, \phi) - g(x) \right| \; dx \; d\nu_{d\times d}(\phi). \tag{3.1}$$

Table 3.1 shows numerical estimations of $\mathcal{S}_d$ for different dimensions. Recall that total variation distance is zero if and only if the measures differ only on measure zero set, and one if they are orthogonal. Since we observe very small $\mathcal{S}_d$, we conclude that for almost any ϕ (except a measure zero set), $f(x, \phi)$ is close to its average $g(x)$. We conclude the distribution of negativity in the post-measurement is independent of the input state.

For two qubit systems, we can derive the distribution of negativity of pure states analytically. The distribution of Schmidt coefficients p_i in a two qubits pure state sampled randomly from the Haar measure is [LP88, Dat10]

$$Pr(p_0, p_1) = 3\,(p_0 - p_1)^2\, \delta(1 - p_0 - p_1). \tag{3.2}$$

Combining the definition of negativity and eq. (3.2), we obtain

$$Pr(N(\psi) = x) = 12x\sqrt{1 - 4x^2}.$$

For higher dimensional systems, we can simply sample random pure states and compute their negativities.

Figure 3.1 shows the distribution of negativity in the post-measurement state, along with the distribution of negativity in random pure states. Observe that the probability of negativity gain goes to zero as the dimensions of the system increase.

Chapter 4

Conclusion

Let us review the results. In chapter 2, we constructed a projection onto a maximally entangled basis that can be implemented by LOCC. We also showed that for two qubit systems, projections in *any* maximally entangled basis has an LOCC implementation. Thus projections along maximally entangled basis are not useful for entanglement creation. We characterized various conditions for entanglement gain in a measurement with unknown outcome. When possible, we gave a state or basis to observe this gain. We showed that while it is almost always possible to find a measurement basis to increase the entanglement of any pure state, for some mixed states it is impossible. We gave partial results on the optimal state/basis pair.

In chapter 3 we studied Markovian dynamics that leads to projections. While it is easy to find a dynamical process that implements a projection in the limit, we showed that it is impossible to have it at a finite time. We also gave numerical evidence explaining the absence of spontaneous entanglement gain due to decoherence in macroscopic systems.

The largest problem left open in this thesis is optimality, that is, how much entanglement can one generate in this fashion? While analysis for a certain family and numerical studies shows that the highest negativity increase is halfway to maximum, we could not prove this analytically. We can also study iterations of this process and derive the ultimate limits of this technique. We suspect that we cannot generate maximally entangled states even in the limit of infinitely many iterations, because in each iteration we can only increase the entropy of the system. This in turn means that the state moves closer to the maximally mixed state. Because there is a ball of absolutely separable state around the maximally mixed state, this suggests the optimal number of iterations is finite. It is unclear if this would perform any better than a single iteration.

It would also be interesting to experimentally observe this kind of gain in entanglement. This will demonstrate yet another pathway to create entanglement.

Bibliography

[Ara80] H. Araki. A remark on machida-namiki theory of measurement. *Progress of Theoretical Physics*, 64(3):719–730, 1980. doi:10.1143/ptp.64.719.

[BO03] Ph. Blanchard and R. Olkiewicz. Decoherence induced transition from quantum to classical dynamics. *Reviews in Mathematical Physics*, 15(03):217–243, 2003. doi:10.1142/s0129055x03001631.

[BW08] R. Blatt and D. Wineland. Entangled states of trapped atomic ions. *Nature*, 453(7198):1008–1015, 2008. doi:10.1038/nature07125.

[Dat10] A. Datta. Negativity of random pure states. *Physical Review A*, 81(5), 2010. doi:10.1103/physreva.81.052312.

[HHHO09] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim. General paradigm for distilling classical key from quantum states. *IEEE Transactions on Information Theory*, 55(4):1898–1929, 2009. doi:10.1109/tit.2008.2009798.

[HM11] U. Haagerup and M. Musat. Factorization and dilation problems for completely positive maps on von neumann algebras. *Communications in Mathematical Physics*, 303(2):555–594, 2011. doi:10.1007/s00220-011-1216-y.

[IH00] S. Ishizaka and T. Hiroshima. Maximally entangled mixed states under nonlocal unitary operations in two qubits. *Physical Review A*, 62(2), 2000. doi:10.1103/physreva.62.022310.

[JKP01] B. Julsgaard, A. Kozhekin, and E. S. Polzik. Experimental long-lived entanglement of two macroscopic objects. *Nature*, 413(6854):400–403, 2001. doi:10.1038/35096524.

[JS17] M. V. Jabir and G. K. Samanta. Robust, high brightness, degenerate entangled photon source at room temperature. *Scientific Reports*, 7(1), 2017. doi:10.1038/s41598-017-12709-5.

[KKS07] M. Khasin, R. Kosloff, and D. Steinitz. Negativity as a distance from a separable state. *Physical Review A*, 75(5), 2007. doi:10.1103/physreva.75.052325.

[KMJ+11] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik. Entanglement generated by dissipation and steady state entanglement of two macroscopic objects. *Physical Review Letters*, 107(8), 2011. doi:10.1103/physrevlett.107.080503.

[LGR+13] Y. Lin, J. P. Gaebler, F. Reiter, T. R. Tan, R. Bowler, A. S. Sørensen, D. Leibfried, and D. J. Wineland. Dissipative production of a maximally entangled steady state of two quantum bits. *Nature*, 504(7480):415–418, 2013. doi:10.1038/nature12801.

[LP88] S. Lloyd and H. Pagels. Complexity as thermodynamic depth. *Annals of Physics*, 188(1):186–213, 1988. URL: https://doi.org/10.1016/0003-4916%2888%2990094-2, doi:10.1016/0003-4916(88)90094-2.

[NC09] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2009. doi:10.1017/cbo9780511976667.

[PTSL13] T. A. Palomaki, J. D. Teufel, R. W. Simmonds, and K. W. Lehnert. Entangling mechanical motion with microwave fields. *Science*, 342(6159):710–713, 2013. doi:10.1126/science.1244563.

[Rai99] E. M. Rains. Bound on distillable entanglement. *Physical Review A*, 60(1):179–184, 1999. doi:10.1103/physreva.60.179.

[RRS16] F. Reiter, D. Reeb, and A. S. Sørensen. Scalable dissipative preparation of many-body entanglement. *Physical Review Letters*, 117(4), 2016. doi:10.1103/physrevlett.117.040501.

[RWM+18] R. Riedinger, A. Wallucks, I. Marinković, C. Löschnauer, M. Aspelmeyer, S. Hong, and S. Gröblacher. Remote quantum entanglement between two micromechanical oscillators. *Nature*, 556(7702):473–477, 2018. doi:10.1038/s41586-018-0036-z.

[SAB+06] M. Steffen, M. Ansmann, R. C. Bialczak, N. Katz, E. Lucero, R. McDermott, M. Neeley, E. M. Weig, A. N. Cleland, and J. M. Martinis. Measurement of the entanglement of two superconducting qubits via state tomography. *Science*, 313(5792):1423–1425, 2006. doi:10.1126/science.1130886.

[SHL+13] S. Shankar, M. Hatridge, Z. Leghtas, K. M. Sliwa, A. Narla, U. Vool, S. M. Girvin, L. Frunzio, M. Mirrahimi, and M. H. Devoret. Autonomously stabilized entanglement between two superconducting quantum bits. *Nature*, 504(7480):419–422, 2013. doi:10.1038/nature12802.

[SVW05] J. A. Smolin, F. Verstraete, and A. Winter. Entanglement of assistance and multipartite state distillation. *Physical Review A*, 72(5), 2005. doi:10.1103/physreva.72.052317.

[Tsy09] Alexandre B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer New York, 2009. doi:10.1007/b13794.

[VAM01] F. Verstraete, K. Audenaert, and B. De Moor. Maximally entangled mixed states of two qubits. *Physical Review A*, 64(1), 2001. doi:10.1103/physreva.64.012316.

[VSPM01] S Virmani, M. F Sacchi, M. B Plenio, and D Markham. Optimal local discrimination of two multipartite pure states. *Physics Letters A*, 288(2):62–68, 2001. doi:10.1016/s0375-9601(01)00484-4.

[VW02] G. Vidal and R. F. Werner. Computable measure of entanglement. *Physical Review A*, 65(3), 2002. doi:10.1103/physreva.65.032314.

[VWC09] F. Verstraete, M. M. Wolf, and J. I. Cirac. Quantum computation and quantum-state engineering driven by dissipation. *Nature Physics*, 5(9):633–636, 2009. doi:10.1038/nphys1342.

[Zur82] W. H. Zurek. Environment-induced superselection rules. *Physical Review D*, 26(8):1862–1880, 1982. doi:10.1103/physrevd.26.1862.