GRAVITY-MEDIATED GAIN OF QUANTUM ENTANGLEMENT

SUBMITTED
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Abstract

No experiment to date provided direct evidence for quantum features of gravitational interaction. Recently proposed tests suggest looking for generation of quantum entanglement between massive objects. Motivated by the success of cooling kilogram mirrors of the LIGO interferometer to near ground state, we study the entanglement dynamics between two massive objects interacting gravitationally. In this thesis, we will focus on two setups: 1) Entanglement of two optomechanical oscillators, 2) Entanglement of two free falling objects. We derive a figure of merit that characterises generated entanglement and entangling time, and show that squeezing of the initial state of the mirrors significantly improves the entanglement. The derivations are supplemented with numerical evidence showing accuracy of our approximations. All this provides a range of experimental parameters required for observation of the gravitationally-induced entanglement.

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Chapter 1

Introduction

1.1 Motivation

The electromagnetic, weak and strong forces are unified wihtin the framework of quantum physics [1, 2]. Naturally one asks about the possibility of unifying the gravitational force under a quantum umbrella. Due to the weakness of the gravitational force, direct observations of effects of gravitational force or couplings prove to be difficult. Several experiments measured the effects of gravity on quantum matters: gravity-induced quantum phase shift in vertical neutron interferometer [3], precise measurement of gravitational acceleration by dropping atoms [4], or quantum bound states of neutrons in a cofining potential created by gravitational field and a horizontal mirror [5] showed successful interactions between quantum particles and a Newtonian gravitational field. Recent experimental scheme to probe inverse square law of Newtonian gravity is described in Ref. [6]. In all these experiment, gravity is treated as classical object. This motivates us to propose a scheme that can truly detect quantum features of gravitational interactions. The basic idea is to look for quantum entanglement between objects mediated by gravitational interaction.

In this paper, we provide two experimental proposals to probe for possible entanglement between spheres. The first proposal involves optomechanical system where the targets of entanglement are kilogram mirrors trapped in harmonic traps, interacting gravitationally. The second proposal involves free falling spheres. A figure of merit characterising both the maximal amount of gravitationally-induced entanglement and the time it takes to observe it are derived analytically. For the case of optomechanical system, the derivation includes various initial states and shows that the mirrors have to be cooled down very close to their ground states and that squeezing of their initial states significantly improves generated entanglement. Numerical simulations of the setups are then performed using current limits of optomechanical experiments, producing a realistic parameters required to observe entanglement.

Entanglement of two quantum objects that are interacting indirectly via a third mediator proves that the mediator is a quantum entity [7, 8]. For our proposal, we do not treat gravity as a mediator, but rather, we attempt to entangle subsystems A and B directly through the gravitational Hamiltonian. We believe that same results are obtained as if we treat gravity as an external mediator. This is due to the fact that gravitational effect propagates at the speed of light, as a result, there is reason to believe gravity is mediated.

1.2 Theoretical frameworks

In this section, we reviewed background information relevant for the projects. We first introduce the continuous variable formalism of Gaussian states and the quantification of entanglement between two Gaussian states. Next, we provide a comprehensive review on squeezed ground states and thermal states.

1.2.1 Gaussian state covariance matrix

The infinite dimensional Hilbert space of quantum states makes numerical calculations difficult and requires dimensional truncation. Using the Wigner-Weyl transformation, one can map Hilbert space operators in the Schrodinger's picture to functions in quantum phase space formulation invertibly, preventing dimensional truncation during calculations.

The Weyl-tranformed function $\tilde{O}$ in quantum phase space formulation for an operator $\hat{O}$ is defined as [9]

$$\tilde{O}(x,p) = \int dy\; e^{-\frac{ipy}{\hbar}} \langle x+\frac{y}{2}|\hat{O}(\hat{x},\hat{p})|x-\frac{y}{2}\rangle\,. \tag{1.1}$$

The trace of two operators can hence be calculated as follows:

$$\begin{aligned}\int_{-\inf}^{inf}\int_{-\infty}^{\infty} dx\; dp\; \tilde{O}_1(x,p)\tilde{O}_2(x,p) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\; dp\; dy\; dy'\; e^{-\frac{ipy}{\hbar}} \langle x+\frac{y}{2}|\hat{O}_1(\hat{x},\hat{p})|x-\frac{y}{2}\rangle \\ &\quad \times e^{-\frac{ipy'}{\hbar}} \langle x+\frac{y'}{2}|\hat{O}_2(\hat{x},\hat{p})|x-\frac{y'}{2}\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\; dp\; dy\; dy'\; e^{-\frac{ip(y+y')}{\hbar}} \langle x+\frac{y}{2}|\hat{O}_1(\hat{x},\hat{p})|x-\frac{y}{2}\rangle \\ &\quad \times \langle x+\frac{y'}{2}|\hat{O}_2(\hat{x},\hat{p})|x-\frac{y'}{2}\rangle\,. \end{aligned} \tag{1.2}$$

Re-expressing the integral of p as a Dirac-delta function,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\; dy'\; f(y')e^{-\frac{ip(y+y')}{\hbar}} &= \int_{-\infty}^{\infty} dy'\; 2\pi\hbar\delta(y+y')f(y') \\ &= 2\pi\hbar f(-y), \end{aligned} \tag{1.3}$$

the integral is then evaluated to be

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\; dp\; \tilde{O}_1(x,p)\tilde{O}_2(x,p) &= 2\pi\hbar \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\; dy\; \langle x+\frac{y}{2}|\hat{O}_1(\hat{x},\hat{p})|x-\frac{y}{2}\rangle \\ &\quad \times \langle x-\frac{y}{2}|\hat{O}_2(\hat{x},\hat{p})|x+\frac{y}{2}\rangle\,. \end{aligned} \tag{1.4}$$

Applying a change in variables $u = x - y/2$ and $v = x + y/2$,

$$\begin{aligned}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dp\ \tilde{O}_1(x,p)\tilde{O}_2(x,p) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\ dv\ \langle v|\hat{O}_1|u\rangle\langle u|\hat{O}_2|v\rangle \\ &= \mathrm{Tr}[\hat{O}_1\hat{O}_2]. \end{aligned} \tag{1.5}$$

The Wigner function is defined as

$$\begin{aligned} W(x,p) &\equiv \frac{\tilde{\rho}}{2\pi\hbar} \quad \text{,where } \tilde{\rho} \text{ is the density matrix} \\ &\equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy\ e^{-\frac{ipy}{\hbar}}\psi\left(x+\frac{y}{2}\right)\psi^*\left(x+\frac{y}{2}\right). \end{aligned} \tag{1.6}$$

The expectation value of an observable can then be expressed in terms of Wigner function

$$\begin{aligned} \langle\hat{O}\rangle &= \mathrm{Tr}[\hat{\rho}\hat{O}] \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dp\ \tilde{\rho}(x,p)\tilde{O}(x,p) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dp\ W(x,p)\tilde{O}(x,p). \end{aligned} \tag{1.7}$$

For symmetrised operator $S(\bar{x}^n\bar{p}^m)$, expressing its expectation value in terms of Wigner function [10],

$$\langle\hat{\rho}S(\hat{x}^n\hat{p}^m)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdpW(x,p)x^np^m. \tag{1.8}$$

The symmetrised operator is defined as such

$$S(\hat{x}^n,\hat{p}^m) = \frac{\hat{x}^n\hat{p} + \hat{x}^{n-1}\hat{p}\hat{x}_+\cdots}{T}, \tag{1.9}$$

where T is the number of numerator terms [11].

For N-mode Gaussian states, their Wigner functions are normalised Gaussian distributions parameterised by N-pairs of position and momentum defining each mode,

$$W(\xi) = \frac{1}{(2\pi)^N\sqrt{\det V^{(N)}}}\exp\left(-\frac{1}{2}\hat{\xi}\left[V^{(N)}\right]^{-1}\hat{\xi}^T\right), \tag{1.10}$$

where $V^{(N)}$ is a $2N \times 2N$ covariance matrix, $\hat{\xi} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_N, \hat{p}_N)$. The Gaussian states are fully described by second moments (i.e. $S(\hat{x},\hat{p}) = (\hat{x}\hat{p} + \hat{p}\hat{x})/2$, where $\hat{x}$ and $\hat{p}$ are position and momentum operators respectively) [12], they are completely imposed by their covariance matrices $V^{(N)}$, which are obtained as

$$\begin{aligned}
\mathrm{Tr}\left[\hat{\rho}\frac{\Delta\hat{\xi}_i\Delta\hat{\xi}_j+\Delta\hat{\xi}_j\Delta\hat{\xi}_i}{2}\right] &= \left\langle\frac{\Delta\hat{\xi}_i\Delta\hat{\xi}_j+\Delta\hat{\xi}_j\Delta\hat{\xi}_i}{2}\right\rangle \\
&= \int_{-\infty}^{\infty} W(\xi)\hat{\xi}_i\hat{\xi}_j\ d^{2N}\hat{\xi} \\
&= V_{ij}^{(N)}, \qquad (1.11)
\end{aligned}$$

where $\Delta\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle$.

1.2.2 Entanglement of bipartite system

The commutation relation for the N-pairs of position and momentum operators $\hat{\xi} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_N, \hat{p}_N)$ describing a Gaussian state can be concisely expressed as a sympletic form Ω,

$$[\hat{\xi}_i, \hat{\xi}_j] = \hat{i}\Omega_{ij} \quad \text{, where } \Omega = \bigoplus_{i=1}^{N}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \hat{i} \text{ refers to imaginary number,} \qquad (1.12)$$

and the elements forming the covariance matrix of a Gaussian state are

$$V_{ij} = \frac{1}{2}\langle\hat{\xi}_i\hat{\xi}_j + \hat{\xi}_j\hat{\xi}_i\rangle - \langle\hat{\xi}_i\rangle\langle\hat{\xi}_j\rangle\,. \qquad (1.13)$$

The constraint for the covariance matrix to represent physical Gaussian state is described by the uncertainty relation [13, 14]

$$2V + \hat{i}\Omega \geq 0. \qquad (1.14)$$

For a two mode Gaussian state, the covariance matrix can be re-written in the form of

$$V = V^{(2)} = \begin{pmatrix} L_{AA} & L_{AB} \\ L_{AB}^T & L_{BB} \end{pmatrix}, \qquad (1.15)$$

where L_{AA} and L_{BB} are 2×2 matrices describing the local mode correlations of the individual mode Gaussian state and L_{AB} describes the intermodal correlations between the two modes.
The constraint for covariance matrix can then be re-expressed in terms of $\det(V)$ and $\sum(V) = \det(L_{AA}) + \det(L_{BB}) + 2\det(L_{AB})$ as [15],

$$\sum(V) \leq 1 + \det(V). \qquad (1.16)$$

An expression for the sympletic eigenvalues of the two-mode Gaussian state is derived to be:

$$v^{\mp} \;=\; (2)^{-1/2}\left(\sum(V) \mp \left(\sum(V)^2 - 4\det(V)\right)^{1/2}\right)^{1/2}, \tag{1.17}$$

with $v^- \leq v^+$, and the uncertainty relation in Eq (1.14) is further simplified to

$$v^- \geq \frac{1}{2}. \tag{1.18}$$

For a two mode system, the Peres-Horodecki positivity of the partially transposed state (PPT) criterion [16] is sufficient to provide an analysis on the separability of the Gaussian states. The partial transposition of a bipartite quantum state is the transposition of either one of the subsystems in a given basis [17]. In our case, we defined the partial transposition with respect to subsystem B, which is equivalent to switching the sign of the momentum operator for the second subsystem in the covariance matrix. The resulting covariance matrix $\tilde{V}$ differs from the original covariance matrix by a sign switch for the determinant of L_{AB}.
The newly defined sympletic eigenvalues are calculated as follows

$$\sum(\tilde{V}) \;=\; \det(L_{AA}) + \det(L_{BB}) - 2\det(L_{AB}) \tag{1.19}$$

$$\Rightarrow \tilde{v}^{\mp} \;=\; (2)^{-1/2}\left(\sum(\tilde{V}) \mp \left(\sum(\tilde{V})^2 - 4\det(\tilde{V})\right)^{1/2}\right)^{1/2}. \tag{1.20}$$

Therefore, based on the PPT criterion of serparability, the two modes are entangled if the minimum sympletic eigenvalue satisfies the inequality

$$\tilde{v}^- \;<\; \frac{1}{2}. \tag{1.21}$$

The quantitative analysis of entanglement can be computed by the logarithmic negativity of the Gaussian states [18, 19]. The negativity of any quantum state ρ' is defined as

$$\mathcal{N}(\rho') \;=\; \frac{\|\tilde{\rho'}\|_1 - 1}{2}, \tag{1.22}$$

where $\|\tilde{\rho'}\|_1 = \mathrm{Tr}|\tilde{\rho'}| = \mathrm{Tr}\sqrt{\tilde{\rho'}^{\dagger}\tilde{\rho'}}$ is the trace norm of the partially transposed density matrix ρ'. The logarithmic negativity $E_{\mathcal{N}}$ is hence related to negativity defined as [18]

$$E_{\mathcal{N}} \;=\; \ln(2\mathcal{N} + 1), \tag{1.23}$$

which establishes an upper bound to the distillable entanglement of the quantum state. For a two-mode Gaussian state ρ', the negativity and the logarithmic negativity are shown to be functions of the minimum sympletic eigenvalue of its partially tranposed state,

$$\|\tilde{\rho}'\|_1 = \frac{1}{2\tilde{v}^-} \quad \Rightarrow \quad \mathcal{N}(\rho') = \max\left[0, \frac{1-2\tilde{v}^-}{4\tilde{v}^-}\right] \tag{1.24}$$

$$\begin{aligned} E_{\mathcal{N}} &= \max\left[0, -\ln(2\mathcal{N}+1)\right] \\ &= \max\left[0, -\ln(2\tilde{v}^-)\right]. \end{aligned} \tag{1.25}$$

Hence, the violation of the PPT criterion for separability, i.e. $\tilde{v}^- < 1/2$, results in a non-zero logarithmic negativity, quantifying the extent of entanglement of the systems [20].

1.2.3 Squeezed ground state

A ground state of quantum harmonic oscillator exhibits a symmetrical distribution of the standard deviations for position and momentum

$$\Delta\bar{x} = \Delta\bar{p} = \sqrt{\frac{1}{2}}, \tag{1.26}$$

where $\bar{x} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$ and $\bar{p} = \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$ are dimensionless position and momentum operators respectively. By applying a squeezing operation , one can reduce one of these standard deviation beyond that of the ground state at the price of increasing the other one.

The ground state wave function of a harmonic oscillator in the position representation is given as

$$\Psi_0(x) = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{x^2}{2}}. \tag{1.27}$$

Performing a Fourier transform, the wave function in momentum representation is

$$\Psi_0(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \Psi_0(x) dx = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{p^2}{2}} \tag{1.28}$$

By squeezing this state with a squeezing factor of $R > 0$, the wave functions of the squeezed states are scaled by a factor of R, where $R = e^{re^{i\theta}}$, with r being the squeezingstrength and θ being the squeezing phase. The resulting wave functions are

$$\Psi_{sq}(x) = \frac{\sqrt{R}}{\pi^{\frac{1}{4}}} e^{-\frac{(Rx)^2}{2}} \tag{1.29}$$

$$\Psi_{sq}(p) = \frac{1}{\sqrt{R}\pi^{\frac{1}{4}}} e^{-\frac{(Rp)^2}{2}}. \tag{1.30}$$

The variances of the position and momentum for the squeezed states are hence calculated to be

$$\langle \Delta \bar{x} \rangle^2 = \frac{1}{2R^2} \tag{1.31}$$

$$\langle \Delta \bar{p} \rangle^2 = \frac{R^2}{2}. \tag{1.32}$$

For position squeezed state (i.e. the position variance is below that of the original ground state), $R > 1$. For momentum squeezed state, $R < 1$.

In matrix formalism, the squeezing operator is [21]

$$\hat{S}(\delta) = exp\left[\frac{\delta^* \hat{a}^2 - \delta \hat{a}^{\dagger 2}}{2}\right], \tag{1.33}$$

where $\delta = re^{i\theta}$ is the squeezing parameter. For ground state squeezing, $r = lnR$ and θ are real numbers.

1.2.4 Thermal state

The distribution of phonons at different energy states of quantum harmonic oscillator for a given temperature conforms to the Bose-Einstein distribution as described in statistical mechanics [22]. Such states are known as thermal states. The density matrix of a thermal state is defined as:

$$\rho_{th} = \frac{1}{Z} e^{-\beta \hat{a}^\dagger \hat{a}} \quad , \beta = \frac{\hbar \omega}{kT}, \tag{1.34}$$

where k is Boltzmann constant, T is the temperature of the state, ω is the frequency of the state and

$$\begin{aligned} Z &= \mathrm{Tr}\{e^{-\beta \hat{a}^\dagger \hat{a}}\} \\ &= \sum_n \langle n| e^{-\beta \hat{a}^\dagger \hat{a}} |n\rangle \\ &= \sum_n e^{-\beta n} \\ &= \frac{1}{1 - e^{-\beta}}. \end{aligned} \tag{1.35}$$

The average number of phonons in the thermal state n of given temperature T is

$$\begin{aligned}
\langle n \rangle &= \mathrm{Tr}\,\langle \rho \hat{n} \rangle \\
&= \mathrm{Tr}\,\langle \frac{1}{Z} e^{-\beta \hat{a}^\dagger \hat{a}} \hat{a}^\dagger \hat{a} \rangle \\
&= \frac{1}{Z} \sum_m \langle m | e^{-\beta \hat{a}^\dagger \hat{a}} \hat{a}^\dagger \hat{a} | m \rangle \\
&= \frac{1}{Z} \sum_m m \, \langle m | e^{-\beta \hat{a}^\dagger \hat{a}} | m \rangle \\
&= \frac{1}{Z} \sum_m m e^{-\beta m} \\
&= \frac{1}{Z} \left[e^{-\beta} + 2e^{-2\beta} + 3e^{-3\beta} + \cdots \right] \\
&= \frac{1}{Z} [e^{-\beta} + e^{-2\beta} + e^{-3\beta} + \cdots \\
&\qquad + e^{-2\beta} + e^{-3\beta} + \cdots \\
&\qquad\qquad + e^{-3\beta} + \cdots] \\
&= \frac{1}{Z} \left[\frac{e^{-\beta}}{1 - e^{-\beta}} + \frac{e^{-2\beta}}{1 - e^{-\beta}} + \frac{e^{-3\beta}}{1 - e^{-\beta}} + \cdots \right] \\
&= \left(1 - e^{-\beta}\right) \left[\frac{1}{1 - e^{-\beta}} \left(\frac{e^{-\beta}}{1 - e^{-\beta}} \right) \right] \\
&= \frac{1}{e^{\beta} - 1}. \qquad (1.36)
\end{aligned}$$

Hence, the density matrix of the thermal state can be re-written as

$$\begin{aligned}
\rho_{th} &= \frac{1}{N+1} \left(\frac{N}{N+1} \right)^{\hat{a}^\dagger \hat{a}} \\
&= \frac{1}{N+1} \sum_{m=0} \left(\frac{N}{N+1} \right)^m |m\rangle \langle m| \,, \qquad (1.37)
\end{aligned}$$

where $N = \langle n \rangle$.

Chapter 2

Entanglement dynamics between two massive oscillators

Consider two spheres of mass m_A and m_B, each with radius R. The two spheres are intially separated by distance L, each trapped in an harmonic trap of frequency ω that also interact via Newtonian gravity as in Figure (3.1).

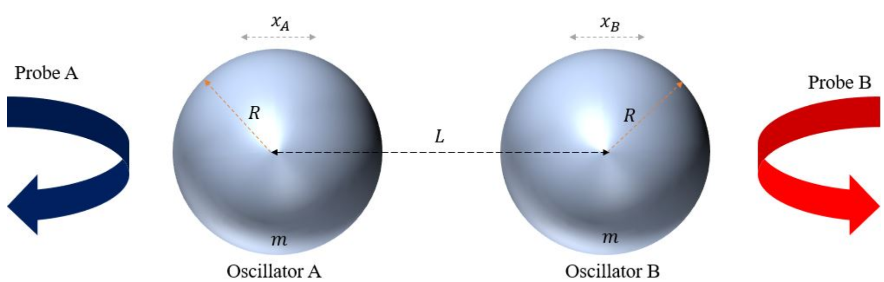


Figure 2.1: Two mechanical spheres trapped within individual harmonic traps interacting purely via gravity. Measurement of the entanglement between the spheres can be done by probing them with weak lights

From Appendix 5.1, we have shown that the interacting Hamiltonian for two spheres due to gravitational interaction is equivalent to that of two point masses. Hence, the Hamiltonian of the proposed system can be expressed as the individual hamiltonian of the oscillators and the interacting gravitational hamiltonian. Approximating the interacting hamiltonian to the second order term, we have:

$$\begin{aligned} \hat{H} &= \frac{\hat{p}_A^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_A^2 + \frac{\hat{p}_B^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_B^2 \\ &\quad -\frac{Gm^2}{L}\left[1 + \frac{(\hat{x}_A - \hat{x}_B)}{L} + \frac{(\hat{x}_A - \hat{x}_B)^2}{L^2}\right]. \end{aligned} \tag{2.1}$$

By expressing the position operators as a linear combination of creation and annihilation operators $\hat{a}_{m,n}, \hat{a}_{m,n}^\dagger$, one can observe that the entanglement coupling arises from the second order expansion of the gravitational interaction. The termination of the expansion of the gravitational Hamiltonian at the second order can be justified by the fact that the standard deviation of the oscillators' position, Δx is much smaller than the separation between the oscillators. Hence, for higher order expansion, $\frac{\Delta x^n}{L^n} \approx 0$. For observable entanglement, the Hamiltonian of coupling term has to be comparable to the individual Hamiltonian of the oscillators. Hence, for ground state oscillators,

$$\begin{aligned} \frac{Gm^2}{L^3}(\hat{x}_A - \hat{x}_B)^2 &\approx \frac{\hbar\omega}{2} \\ \frac{Gm}{\omega^2 L^3} &\approx 1, \quad \text{where } (\hat{x}_A - \hat{x}_B) = \sqrt{\frac{\hbar}{2m\omega}}. \end{aligned} \tag{2.2}$$

We hence define a figure of merit $\eta = \frac{2Gm}{\omega^2 L^3}$ as the relevant factor for entanglement. Remarkably, this simple figure of merit characterises both maximal entanglement achievable within the considered system and time required to achieve it. The derivations below will give all the details.

From the Hamiltonian, the equations of motion for the system in Heisenberg picture are derived as follows

$$\begin{aligned} \dot{\hat{x}}_A &= \frac{i}{\hbar}[\hat{H}, \hat{x}_A] + \frac{\partial \hat{x}_A}{\partial t} \\ &= \frac{i}{\hbar}\left(\frac{\hat{p}_A^2}{2m}\hat{x}_A - \hat{x}_A\frac{\hat{p}_A^2}{2m}\right) \\ &= \frac{i}{\hbar}\left(\frac{-i\hbar}{m}\hat{p}_A\right) = \frac{\hat{p}_A}{m} \\ \dot{\bar{x}}_A &= \omega\bar{p}_A \end{aligned} \tag{2.3}$$

$$\begin{aligned} \dot{\hat{p}}_A &= \frac{i}{\hbar}[\hat{H}, \hat{p}_A] + \frac{\partial \hat{p}_A}{\partial t} \\ &= \frac{i}{\hbar}\left[-m\omega^2\frac{\hbar}{i}\hat{x}_A + \frac{Gm^2}{L^2}\frac{\hbar}{i} + \frac{Gm^2}{L^3}\frac{\hbar}{i}2(\hat{x}_A - \hat{x}_B)\right] \\ \dot{\bar{p}}_A &= \left(\frac{2Gm}{\omega L^3} - \omega\right)\bar{x}_A - \frac{2Gm}{wL^3}\bar{x}_B + \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \dot{\hat{x}}_B &= \frac{i}{\hbar}[\hat{H}, \hat{x}_B] + \frac{\partial \hat{x}_B}{\partial t} \\ &= \frac{i}{\hbar}\left(\frac{\hat{p}_B^2}{2m}\hat{x}_B - \hat{x}_A\frac{\hat{p}_B^2}{2m}\right) \\ &= \frac{i}{\hbar}\left(\frac{-i\hbar}{m}\hat{p}_B\right) = \frac{\hat{p}_B}{m} \\ \dot{\bar{x}}_B &= \omega\bar{p}_B \end{aligned} \tag{2.5}$$

$$\begin{aligned}\dot{\hat{p}}_B &= \frac{i}{\hbar}[\hat{H},\hat{p}_B] + \frac{\partial \hat{p}_B}{\partial t} \\ &= \frac{i}{\hbar}\left[-m\omega^2\frac{\hbar}{i}\hat{x}_B - \frac{Gm^2}{L^2}\frac{\hbar}{i} - \frac{Gm^2}{L^3}\frac{\hbar}{i}2(\hat{x}_A - \hat{x}_B)\right] \\ \dot{\bar{p}}_B &= \left(\frac{2Gm}{\omega L^3} - \omega\right)\bar{x}_B - \frac{2Gm}{wL^3}\bar{x}_A - \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}}. \end{aligned} \tag{2.6}$$

The position and momentum operators $(\hat{x},\hat{p})$ are converted to dimensionless forms $(\bar{x},\bar{p})$. The above equations can be re-written in a matrix form $\dot{u}(t) = Ku(t) + c$,

$$\begin{pmatrix}\dot{\bar{x}}_A \\ \dot{\bar{p}}_A \\ \dot{\bar{x}}_B \\ \dot{\bar{p}}_B\end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \left(\frac{2Gm}{\omega L^3} - \omega\right) & 0 & -\frac{2Gm}{\omega L^3} & 0 \\ 0 & 0 & 0 & \omega \\ -\frac{2Gm}{\omega L^3} & 0 & \left(\frac{2Gm}{\omega L^3} - \omega\right) & 0 \end{pmatrix} \begin{pmatrix}\bar{x}_A \\ \bar{p}_A \\ \bar{x}_B \\ \bar{p}_B\end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}} \\ 0 \\ -\frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}}\end{pmatrix}. \tag{2.7}$$

The solution is

$$u(t) = e^{Kt}u(0) + e^{Kt}\int_0^t ce^{-Kt'}dt', \tag{2.8}$$

and since the equations of motion are linear, the dynamics of the system is gaussianity-preserving. As such, we can characterise the system at all times using covariance matrix defined by $V_{ij}(0) = \frac{1}{2}\langle\{\Delta u_i(0), \Delta u_j(0)\}\rangle$, where $\{.,.\}$ is the anti-commutator and $\Delta u = u - \langle u\rangle$

$$\begin{aligned} V_{ij}(0) &= \frac{1}{2}\langle\{\Delta u_i(0), \Delta u_j(0)\}\rangle \\ &= \frac{1}{2}\langle(u_i(0) - \langle u_i(0)\rangle)(u_j(0) - \langle u_j(0)\rangle) + (u_j(0) - \langle u_j(0)\rangle)(u_i(0) - \langle u_i(0)\rangle)\rangle \\ &= \frac{1}{2}(\langle u_i(0)u_j(0)\rangle - 2\langle u_i(0)\rangle\langle u_j(0)\rangle + \langle u_j(0)u_i(0)\rangle) \\ &= \frac{1}{2}\langle u_i(0)u_j(0) + u_j(0)u_i(0)\rangle - \langle u_i(0)\rangle\langle u_j(0)\rangle. \end{aligned} \tag{2.9}$$

The elements of the time dependent covariance matrix are derived as $V_{ij}(t) = \frac{1}{2}\langle\{\Delta u_i(t), \Delta u_j(t)\}\rangle$. Note that

$$\frac{1}{2}\{\Delta u_i(t), \Delta u_j(t)\} = \frac{1}{2}\langle u_i(t)u_j(t) + u_j(t)u_i(t)\rangle - \langle u_i(t)\rangle\langle u_j(t)\rangle. \tag{2.10}$$

Evaluating $\Delta u(t)$ using Eq (2.8),

$$\begin{aligned}
\Delta u(t) &= u(t) - \langle u(t) \rangle \\
&= e^{Kt}u(0) + e^{Kt}\int_0^t ce^{-Kt'}dt' - \left\langle e^{Kt}u(0) + e^{Kt}\int_0^t ce^{-Kt'}dt' \right\rangle \\
&= e^{Kt}u(0) - \langle e^{Kt}u(0) \rangle + e^{Kt}\int_0^t ce^{-Kt'}dt' - \left\langle e^{Kt}\int_0^t ce^{-Kt'}dt' \right\rangle \\
&= e^{Kt}u(0) - \langle e^{Kt}u(0) \rangle . \qquad (2.11)
\end{aligned}$$

As such, we can ignore the term with constant c (see Eq (2.8)) as it does not contribute to $\Delta u(t)$, and we can rewrite time dependent dynamics of the systems as $u(t) = e^{Kt}u(0) = M(t)u(0)$,

$$\begin{aligned}
V_{ij}(t) &= \frac{1}{2}\left\langle \sum_k M_{ik}(t)u_k(0)\sum_l M_{jl}(t)u_l(0) + \sum_l M_{jl}(t)u_l(0)\sum_k M_{ik}(t)u_k(0) \right\rangle \\
&\quad - \left\langle \sum_k M_{ik}(t)u_k(0) \right\rangle \left\langle \sum_l M_{jl}(t)\langle u_l(0)\rangle \right\rangle \\
&= \sum_{kl} M_{ik}(t)M_{jl}(t)\left[\frac{1}{2}\left(\langle u_k(0)u_l(0) + u_l(0)u_k(0)\rangle\right) - \langle u_k(0)\rangle\langle u_l(0)\rangle\right] \\
&= \sum_{kl} M_{ik}(t)M_{jl}(t)V_{kl}(0) \\
&= \sum_{kl} M_{ik}(t)V_{kl}(0)M_{lj}(t)^T . \qquad (2.12)
\end{aligned}$$

Hence, the time dependent covariance matrix is

$$\begin{aligned}
V(t) &= M(t)V(0)M(t)^T \\
&= e^{Kt}V(0)\left(e^{Kt}\right)^T . \qquad (2.13)
\end{aligned}$$

From the equations of motions, K is the matrix defined as

$$K = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \frac{2Gm}{\omega L^3} - \omega & 0 & -\frac{2Gm}{\omega L^3} & 0 \\ 0 & 0 & 0 & \omega \\ -\frac{2Gm}{\omega L^3} & 0 & \frac{2Gm}{\omega L^3} - \omega & 0 \end{pmatrix}. \qquad (2.14)$$

Let $e^{Kt} = e^{AT}$, where $A = K/\omega$, T is a dimensionless time variable ωt, then

$$AT = \begin{pmatrix} 0 & T & 0 & 0 \\ \eta T - T & 0 & -\eta T & 0 \\ 0 & 0 & 0 & T \\ -\eta T & 0 & \eta T - T & 0 \end{pmatrix}. \qquad (2.15)$$

Solving for eigenvalues of matrix AT:

$$\begin{aligned}
\det(AT - \lambda\mathbb{1}) &= 0 \\
\det\begin{pmatrix} -\lambda & T & 0 & 0 \\ \eta T - T & -\lambda & -\eta T & 0 \\ 0 & 0 & -\lambda & T \\ -\eta T & 0 & \eta T - T & -\lambda \end{pmatrix} &= 0 \\
\lambda^4 + \lambda^2(2T^2 - 2\eta T^2) + (T^4 - 2\eta T^4) &= 0 \\
\frac{-(2T^2 - 2\eta T^2) \pm \sqrt{(2T^2 - 2\eta T^2)^2 - 4(T^4 - 2\eta T^4)}}{2} &= \lambda^2 \\
\lambda_{1,2}^2 = \eta T^2 - \eta T^2 - T^2 \quad &\text{or} \quad \lambda_{3,4}^2 = \eta T^2 + \eta T^2 - T^2 \\
\lambda_{1,2} = \pm iT \quad &\text{or} \quad \lambda_{3,4} = \pm T\sqrt{2\eta - 1}.
\end{aligned} \tag{2.16}$$

Solving for eigenvectors of matrix At:

$$v_1 = \begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix} v_2 = \begin{pmatrix} i \\ 1 \\ i \\ 1 \end{pmatrix} v_3 = \begin{pmatrix} \frac{1}{\sqrt{2\eta-1}} \\ -1 \\ -\frac{1}{\sqrt{2\eta-1}} \\ 1 \end{pmatrix} v_4 = \begin{pmatrix} -\frac{1}{\sqrt{2\eta-1}} \\ -1 \\ \frac{1}{\sqrt{2\eta-1}} \\ 1 \end{pmatrix}. \tag{2.17}$$

Using the Fundamental Matrix Solution stated in Appendix 5.2.1,

$$e^{TA} = C_1 e^{iT} \begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix} + C_2 e^{-iT} \begin{pmatrix} i \\ 1 \\ i \\ 1 \end{pmatrix} + C_3 e^{-T\sqrt{2\eta-1}} \begin{pmatrix} \frac{1}{\sqrt{2\eta-1}} \\ -1 \\ -\frac{1}{\sqrt{2\eta-1}} \\ 1 \end{pmatrix} + C_4 e^{T\sqrt{2\eta-1}} \begin{pmatrix} -\frac{1}{\sqrt{2\eta-1}} \\ -1 \\ \frac{1}{\sqrt{2\eta-1}} \\ 1 \end{pmatrix}. \tag{2.18}$$

The fundamental matrix solution is defined to be

$$M(T) = \begin{pmatrix} -ie^{iT} & ie^{-iT} & \frac{1}{\sqrt{2\eta-1}}e^{-T\sqrt{2\eta-1}} & -\frac{1}{\sqrt{2\eta-1}}e^{T\sqrt{2\eta-1}} \\ e^{iT} & e^{-iT} & -e^{-T\sqrt{2\eta-1}} & -e^{T\sqrt{2\eta-1}} \\ -ie^{iT} & ie^{-iT} & -\frac{1}{\sqrt{2\eta-1}}e^{-T\sqrt{2\eta-1}} & \frac{1}{\sqrt{2\eta-1}}e^{T\sqrt{2\eta-1}} \\ e^{iT} & e^{-iT} & e^{-T\sqrt{2\eta-1}} & e^{T\sqrt{2\eta-1}} \end{pmatrix} \tag{2.19}$$

$$M(0)^{-1} = \begin{pmatrix} -\frac{1}{4i} & \frac{1}{4} & -\frac{1}{4i} & \frac{1}{4} \\ \frac{1}{4i} & \frac{1}{4} & \frac{1}{4i} & \frac{1}{4} \\ \frac{\sqrt{2\eta-1}}{4} & -\frac{1}{4} & -\frac{\sqrt{2\eta-1}}{4} & \frac{1}{4} \\ -\frac{\sqrt{2\eta-1}}{4} & -\frac{1}{4} & \frac{\sqrt{2\eta-1}}{4} & \frac{1}{4} \end{pmatrix} \tag{2.20}$$

$$\begin{aligned} e^{TA} &= M(T)M(0)^{-1} \\ &= \frac{1}{2}\begin{pmatrix} \cos T + \cos(T\sqrt{1-2\eta}) & \sin T + \frac{\sin(T\sqrt{1-2\eta}}{\sqrt{1-2\eta}}) & \cos T - \cosh(T\sqrt{1-2\eta}) & \sin T - \frac{\sin(T\sqrt{1-2\eta})}{\sqrt{1-2\eta}} \\ -\sin T - \sqrt{1-2\eta}\sin(T\sqrt{1-2\eta}) & \cos T + \cos(T\sqrt{1-2\eta}) & -\sin T + \sqrt{1-2\eta}\sin(T\sqrt{1-2\eta}) & \cos T - \cos(T\sqrt{1-2\eta}) \\ \cos T - \cos(T\sqrt{1-2\eta}) & \sin T - \frac{\sin(T\sqrt{1-2\eta})}{\sqrt{1-2\eta}} & \cos T + \cos(T\sqrt{1-2\eta}) & \sin T + \frac{\sin(T\sqrt{1-2\eta})}{\sqrt{1-2\eta}} \\ -\sin T + \sqrt{1-2\eta}\sin(T\sqrt{1-2\eta}) & \cos T - \cos(T\sqrt{1-2\eta}) & -\sin T - \sqrt{1-2\eta}\sin(T\sqrt{1-2\eta}) & \cos T + \cos(T\sqrt{1-2\eta}) \end{pmatrix} \end{aligned} \tag{2.21}$$

To simplify the matrix calculation, we let $\varepsilon = \sqrt{1-2\eta}$, $a = \cos T$, $b = \cos(\varepsilon T)$, $c = \sin T$, $d = \sin(\varepsilon T)$,

$$e^{TA} = \frac{1}{2}\begin{pmatrix} a+b & c+\frac{d}{\varepsilon} & a-b & c-\frac{d}{\varepsilon} \\ -c-d\varepsilon & a+b & -c+d\varepsilon & a-b \\ a-b & c-\frac{d}{\varepsilon} & a+b & c+\frac{d}{\varepsilon} \\ -c+d\varepsilon & a-b & -c-d\varepsilon & a+b \end{pmatrix}. \tag{2.22}$$

2.1 Ground state

2.1.1 Derivation of covariance matrix

Consider two quantum harmonic oscillators initially prepared in ground states. The evaluation for the entanglement evolution of the system of the system first requires solving the covariance matrix of the systems for ground state harmonic oscillators at time ωt.

We first begin by deriving the covariance matrix of the initial state of the system. For intermodal correlation, the single mode operators act on different subsystem, resulting in the reduction of those terms to zero. Hence, the resulting non-zero terms are local mode correlation for the respective subsystem. The full derivation can be found in Appendix 5.3.1

The covariance matrix describing the initial state of the system is hence

$$V_{gd}(0) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{2.23}$$

We can next calculate the covariance matrix at time t,

$$\begin{aligned} V_{gd}(t) &= \frac{1}{4}\begin{pmatrix} a+b & c+\frac{d}{\varepsilon} & a-b & c-\frac{d}{\varepsilon} \\ -c-d\varepsilon & a+b & -c+d\varepsilon & a-b \\ a-b & c-\frac{d}{\varepsilon} & a+b & c+\frac{d}{\varepsilon} \\ -c+d\varepsilon & a-b & -c-d\varepsilon & a+b \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} a+b & -c-d\varepsilon & a-b & -c+d\varepsilon \\ c+\frac{d}{\varepsilon} & a+b & c-\frac{d}{\varepsilon} & a-b \\ a-b & -c+d\varepsilon & a+b & -c-d\varepsilon \\ c-\frac{d}{\varepsilon} & a-b & c+\frac{d}{\varepsilon} & a+b \end{pmatrix} \\ &= \begin{pmatrix} L_{AA} & L_{AB} \\ L_{AB}^T & L_{BB} \end{pmatrix}. \end{aligned} \tag{2.24}$$

The elements for the L_{AA} are

$$\begin{aligned} V_{11} &= \frac{1}{8}\left((a+b)^2 + \left(c+\frac{d}{\varepsilon}\right)^2 + (a-b)^2 + \left(c-\frac{d}{\varepsilon}\right)\right) \\ V_{12} &= \frac{1}{8}\left((a+b)(-c-d\varepsilon) + (a+b)\left(c+\frac{d}{\varepsilon}\right) + (a-b)(-c+d\varepsilon) + (a-b)\left(c-\frac{d}{\varepsilon}\right)\right) \\ V_{21} &= \frac{1}{8}\left((a+b)(-c-d\varepsilon) + (a+b)\left(c+\frac{d}{\varepsilon}\right) + (a-b)(-c+d\varepsilon) + (a-b)\left(c-\frac{d}{\varepsilon}\right)\right) \\ V_{22} &= \frac{1}{8}\left((-c-d\varepsilon)^2 + (a+b)^2 + (-c+d\varepsilon)^2 + (a-b)^2\right). \end{aligned} \tag{2.25}$$

The elements for the L_{AB} are

$$
\begin{aligned}
V_{13} &= \frac{1}{8}\left((a+b)(a-b)+\left(c+\frac{d}{\varepsilon}\right)\left(c-\frac{d}{\varepsilon}\right)\right)\\
V_{14} &= \frac{1}{8}\left((a+b)(-c+d\varepsilon)+(a-b)\left(c+\frac{d}{\varepsilon}\right)+(a-b)(-c-d\varepsilon)+(a+b)\left(c-\frac{d}{\varepsilon}\right)\right)\\
V_{23} &= \frac{1}{8}\left((a-b)(-c-d\varepsilon)+(a+b)\left(c-\frac{d}{\varepsilon}\right)+(a+b)\,(-c+d\varepsilon)+(a-b)\left(c+\frac{d}{\varepsilon}\right)\right)\\
V_{24} &= \frac{1}{8}\left((-c-d\varepsilon)(-c+d\varepsilon)+(a-b)(a+b)\right).
\end{aligned}
\tag{2.26}
$$

The elements for the L_{BB} are

$$
\begin{aligned}
V_{33} &= \frac{1}{8}\left((a-b)^2+e^{-2r_A}\left(c-\frac{d}{\varepsilon}\right)^2+(a+b)^2+\left(c+\frac{d}{\varepsilon}\right)\right)\\
V_{34} &= \frac{1}{8}\left((a-b)(-c+d\varepsilon)+(a-b)\left(c-\frac{d}{\varepsilon}\right)+(a+b)(-c-d\varepsilon)+(a+b)\left(c+\frac{d}{\varepsilon}\right)\right)\\
V_{43} &= \frac{1}{8}\left((a-b)(-c+d\varepsilon)+(a-b)\left(c-\frac{d}{\varepsilon}\right)+(a+b)(-c-d\varepsilon)+(a+b)\left(c+\frac{d}{\varepsilon}\right)\right)\\
V_{44} &= \frac{1}{8}\left((-c+d\varepsilon)^2+(a-b)^2+(-c-d\varepsilon)^2+(a+b)^2\right).
\end{aligned}
\tag{2.27}
$$

2.1.2 Analytical derivation of maximum entanglement

Using logarithmic negativity as the quantifier for entanglement,

$$
E_{A:B}(t) = [0, -\ln(2\tilde{v}^-)] \tag{2.28}
$$

$$
\tilde{v}^- = 2^{-1/2}\sqrt{\sum(V_{gd})-\sqrt{\sum(V_{gd})^2-4\det(V_{gd})}}, \tag{2.29}
$$

where $\sum(V_{gd}) = \det L_{AA} + \det L_{BB} - 2\det L_{AB}$.

$$
\begin{aligned}
\det(V_{gd}) &= \frac{a^4b^4}{16}+\frac{a^2d^4}{16}+\frac{b^4d^4}{16}+\frac{c^4d^4}{16}+\frac{a^4b^2d^2}{8}+\frac{b^c c^4 d^2}{8}+\frac{a^2b^4c^2}{8}+\frac{a^2c^2d^4}{8}+\frac{a^2b^2c^2d^2}{8}\\
&= \frac{\cosh^4(\varepsilon T)\cos^4 T+\cos^2 T\sinh^4(\varepsilon T)+\cosh^4(\varepsilon T)\sin^4(T)+sinh^4(\varepsilon T)\sin^4(\varepsilon T)}{16}\\
&\quad+\frac{\cosh^2(\varepsilon T)\sinh^2(\varepsilon T)(\cos^4 T+\sin^4 T)}{8}+\frac{\cos^2 T\sin^2 T(\cosh^4(\varepsilon T)+\sinh^4(\varepsilon T))}{8}\\
&\quad+\frac{\cosh^2(\varepsilon T)\sinh^2(\varepsilon T)\cos^2 T\sin^2 T}{4}\\
&= \frac{1}{16}
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
\sum(V_{gd}) &= \frac{1}{16\varepsilon^2}\Big[2c^2d^2 + 8a^2b^2\varepsilon^2 + 4c^2d^2\varepsilon^2 + 2c^2d^2\varepsilon^4 + 4a^2d^2 + c^2d^2 + c^2d^2 + 4b^2c^2\varepsilon^2 + 4b^2c^2\varepsilon^2 \\
&\quad +4a^2d^2\varepsilon^4 - 2c^2d^2\varepsilon^2 - 2c^2d^2\varepsilon^2 + c^2d^2\varepsilon^4 + c^2d^2\varepsilon^4 + 8abcd\varepsilon + 8abcd\varepsilon^3 - 8abcd\varepsilon^3 - 8abcd\varepsilon\Big] \\
&= \frac{1}{64}\left[8\left(2ab + cd\left(\varepsilon + \frac{1}{\varepsilon}\right)\right)^2 + 16\left(\frac{ad}{\varepsilon} - bc\right)^2 + 4c^2d^2\left(\varepsilon - \frac{1}{\varepsilon}\right)^2 \right. \\
&\quad \left. +16(bc - ad\varepsilon)^2 + 4c^2d^2\left(\varepsilon - \frac{1}{\varepsilon}\right)^2\right]. \qquad (2.31)
\end{aligned}
$$

From the definition of η, we note that it is typically small. We therefore Taylor expand ε about $\eta = 0$ to the second order. Substituting the following approximation, $\frac{1}{\varepsilon^2} \approx 1 + 2\eta + 4\eta^2$ and $\varepsilon \approx 1 - \eta - \frac{\eta^2}{2}$, $\sum(V)$ simplifies to

$$
\begin{aligned}
\sum(V_{gd}) &= \frac{1}{64}\left[16\varepsilon^2c^2d^2 + 32b^2c^2 + \frac{16c^2d^2 + 16a^2d^2}{\varepsilon^2} + 32a^2b^2 + 16\varepsilon^2a^2d^2\right] \\
&= \frac{1}{64}\left[16(1-2\eta)c^2d^2 + 32b^2c^2 + (16c^2d^2 + 16a^2d^2)(1 + 2\eta + 4\eta) + 32a^2b^2 + 16(1-2\eta)a^2d^2\right] \\
&= \frac{1}{64}\left[32d^2(a^2 + c^2) + 64\eta^2d^2(a^2 + c^2) + 32b^2(a^2 + c^2)\right] \\
&= \frac{1}{64}\left[(\sin^2 t + \cos^2 t)(32d^2 + 64\eta^2d^2 + 32b^2)\right] \\
&= \frac{1}{64}\left[32d^2 + 64\eta^2d^2 + 32b^2\right] \\
&= \frac{1}{2}\left[\cos^2(\varepsilon T) + \sin^2(\varepsilon T) + 2\eta^2\sin^2(\varepsilon T)\right] \\
&\approx \frac{1}{2}\left[1 + 2\eta^2\sin^2\left(\left(1 - \eta - \frac{\eta^2}{2}\right)T\right)\right]. \qquad (2.32)
\end{aligned}
$$

The maximal entanglement achievable for this setup can hence be derived. In order to acheive maximum entanglement, $\sum(V_{gd})$ has to be maximised. This correlation is proven in Appendix 5.4.
To achieve maximum $\sum(V_{gd})$, $\sin^2\left(\left(1 - \eta - \frac{\eta^2}{2}\right)T_{max}\right) = 1$,

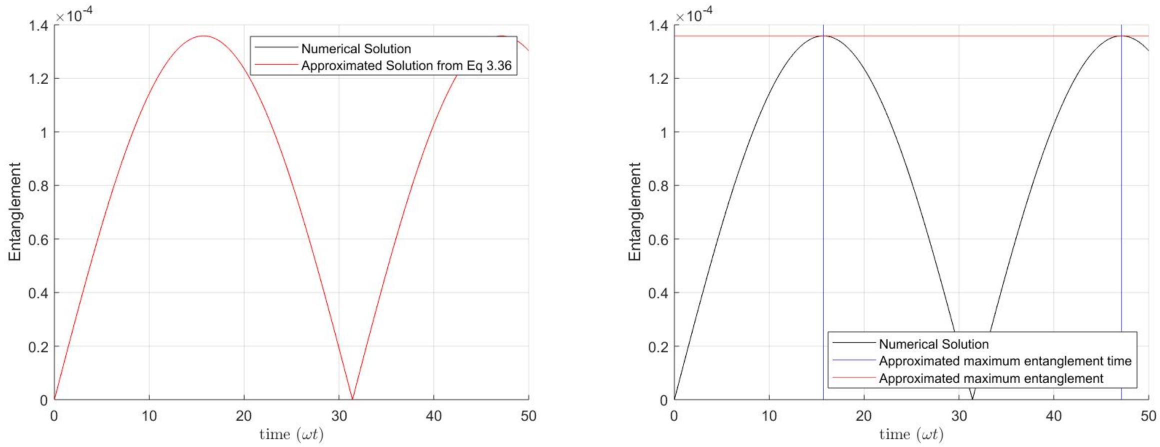
$$
\begin{aligned}
\sin\left(\left(1 - \eta - \frac{\eta^2}{2}\right)T_{max}\right) &= 1 \\
T_{max} &\approx \frac{\pi\left(n + \frac{1}{2}\right)}{1 - \eta} \quad , n \in \mathbb{Z} \qquad (2.33) \\
& \qquad (2.34)
\end{aligned}
$$

$$
\begin{aligned}
\sum (V_{gd})_{max} &= \frac{1}{2}(1+2\eta^2) && (2.35)\\
\tilde{v}^-_{max} &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(1+2\eta^2) - \left[\left(\frac{1}{2}(1+2\eta^2)\right)^2 - \frac{1}{4}\right]^{1/2}\right]^{1/2}\\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{2}+\eta^2-(4\eta^2+4\eta^4)^{1/2}\right]^{1/2}\\
&\approx \frac{1}{\sqrt{2}}\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]^{1/2} && (2.36)\\
E^{A:B}_{max} &= -\ln(2\tilde{v}^-_{max})\\
&\approx -\ln\left[2\left\{\frac{1}{2}\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]\right\}^{1/2}\right]\\
&= -\ln\left[4\left\{\frac{1}{2}\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]\right\}\right]^{1/2}\\
&= -\frac{1}{2}\ln\left[2\left(\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right)\right]\\
&= -\frac{1}{2}\ln[1-2\eta+2\eta^2-\eta^3]\\
&\approx -\frac{1}{2}\ln(1-2\eta)\\
&\approx \eta. && (2.37)
\end{aligned}
$$

The maximum achievable entanglement is equivalent to the figure of merit η defined earlier as the relevant term for coupling.

2.1.3 Computational analysis

Gravitational waves detection setup at LIGO provides kilogram mass mirror cooled to near quantum ground state [23]. As such, it is reasonable to propose the parameters of mirrors from LIGO as possible harmonic oscillators for our experimental setup. We used osmium as a material to maximise the gravitational interactions due to its high density.



(a) Comparison of approximated solution and exact solution for entanglement dynamics

(b) Entanglement dynamics with approximated maximal entanglement value and time from Section 2.1.2

Figure 2.2: Entanglement (Logarithmic negativity) dynamics of identical osmium spheres with parameters: $m = 1\text{kg}$, $\rho = 22500\text{kg/m}^3$, $\omega = 0.1\text{Hz}$, $L = 2.1R$

Figure (2.2b) plots the entanglement dynamics for exact solution calculated directly from the covariance matrix given by Eq (3.31) and the approximated solution derived from Section 3.1.2. The negligible difference in the absolute values of the logarithmic negativity shows that appropriate approximations were made and the derived analytical solutions are accurate in the limits of given parameters. Hence, the maximal entanglement achievable for ground state harmonic oscillator when starting with the ground state harmonic oscillator is

$$E_{max}^{A:B} \quad = \quad \eta \ = \ 1.36 \times 10^{-4}, \tag{2.38}$$

and the time required to accumulate maximal entanglement is

$$T_{max} \quad = \quad \frac{0.5\pi}{1-\eta} \ = \ 1.57 \tag{2.39}$$

$$t_{max} \quad = \quad \frac{1.57}{\omega} \ = \ 15.7s. \tag{2.40}$$

From current optomechanical experiment [24], logarithmic negativity in the order of $\sim 10^{-2}$ has been observed. Since this is two orders of magnitude higher than our predicted entanglement (Eq (2.38)), we now propose squeezing as a way to improve on entanglement.

2.2 Squeezed ground state

Squeezing the harmonic oscillators in either the position or momentum space, we increase the spread of the wavefunctions. This reduces the interaction distance between the oscillators, increasing gravitational interaction. The resulting entanglement between the two subsystems increases accordingly.

2.2.1 Derivation of covariance matrix

The density matrix of squeezed ground state is initialised as $\rho_{sq} = \hat{S}\rho\hat{S}^\dagger$, where $\rho = |0\rangle\langle 0|$ is the ground state density matrix.

For any operator, the expectation value of the operator for squeezed ground state is

$$\begin{aligned}\langle\hat{O}\rangle &= \text{Tr}[\hat{S}\rho\hat{S}^\dagger\hat{O}] \\ &= \text{Tr}[\hat{S}^\dagger\hat{O}\hat{S}\rho] \quad ,\because \text{cyclical properties of trace.}\end{aligned} \tag{2.41}$$

Hence, the expectation value of the following operators are re-expressed as

$$\begin{aligned}\langle\bar{x}\rangle &= \text{Tr}[\hat{S}^\dagger\bar{x}\hat{S}\rho] \\ &= \text{Tr}\left[\hat{S}^\dagger\left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)\hat{S}\rho\right]\end{aligned} \tag{2.42}$$

$$\begin{aligned}\langle\bar{p}\rangle &= \text{Tr}[\hat{S}^\dagger\bar{x}\hat{S}\rho] \\ &= \text{Tr}\left[\hat{S}^\dagger\left(\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}}\right)\hat{S}\rho\right]\end{aligned} \tag{2.43}$$

$$\begin{aligned}\langle\bar{x}^2\rangle &= \text{Tr}[\hat{S}^\dagger\bar{x}^2\hat{S}\rho] \\ &= \text{Tr}\left[\hat{S}^\dagger\left(\frac{\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}}{2}\right)\hat{S}\rho\right]\end{aligned} \tag{2.44}$$

$$\begin{aligned}\langle\bar{p}^2\rangle &= \text{Tr}[\hat{S}^\dagger\bar{p}^2\hat{S}\rho] \\ &= \text{Tr}\left[\hat{S}^\dagger\left(\frac{\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}}{-2}\right)\hat{S}\rho\right].\end{aligned} \tag{2.45}$$

Using Baker-Campbell-Hausdorff formula, we resolve the relationship between the squeezing operators, creation and annihilation operators

$$\begin{aligned} \hat{S}^{\dagger}\hat{a}\hat{S} &= \hat{a}\cosh(r) - \hat{a}^{\dagger}e^{i\phi}\sinh(r) && (2.46) \\ \hat{S}\hat{a}\hat{S}^{\dagger} &= \hat{a}\cosh(r) + \hat{a}^{\dagger}e^{i\phi}\sinh(r) && (2.47) \\ \hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S} &= \hat{a}^{\dagger}\cosh(r) - \hat{a}e^{-i\phi}\sinh(r) && (2.48) \\ \hat{S}\hat{a}^{\dagger}\hat{S}^{\dagger} &= \hat{a}^{\dagger}\cosh(r) + \hat{a}e^{-i\phi}\sinh(r). && (2.49) \end{aligned}$$

Substituting these relations back to the exectation values,

$$\begin{aligned} \langle\bar{x}\rangle &= 0 && (2.50) \\ \langle\bar{p}\rangle &= 0 && (2.51) \\ \langle\bar{x}^2\rangle &= \frac{1}{2}(\cosh(2r) - \sinh(2r\cos(\phi))) && (2.52) \\ \langle\bar{p}^2\rangle &= \frac{1}{2}(\cosh(2r) + \sinh(2r\cos(\phi))) && (2.53) \\ \langle\bar{x}\bar{p} + \bar{p}\bar{x}\rangle &= -\cosh(r)\sinh(r)\sin(\phi). && (2.54) \end{aligned}$$

Full derivations can be found in Appendix 5.3.2. In order to increase the interaction between the two subsystems, we can increase the position uncertainties by squeezing the harmonic oscillators in the momentum basis (i.e. $\phi_A = \phi_B = \pi$), the resulting covariance matrix is:

$$V_{sq}(0) = \frac{1}{2}\begin{pmatrix} e^{2r_A} & 0 & 0 & 0 \\ 0 & e^{-2r_A} & 0 & 0 \\ 0 & 0 & e^{2r_B} & 0 \\ 0 & 0 & 0 & e^{-2r_B} \end{pmatrix}. \qquad (2.55)$$

Refer to Appendix 5.3.2 for derivation of covariance matrix for arbitrary ϕ_A and ϕ_B We can hence calculate the covariance matrix at time T,

$$\begin{aligned} V_{sq}(t) &= \frac{1}{8}\begin{pmatrix} a+b & c+\frac{d}{\varepsilon} & a-b & c-\frac{d}{\varepsilon} \\ -c-d\varepsilon & a+b & -c+d\varepsilon & a-b \\ a-b & c-\frac{d}{\varepsilon} & a+b & c+\frac{d}{\varepsilon} \\ -c+d\varepsilon & a-b & -c-d\varepsilon & a+b \end{pmatrix}\begin{pmatrix} e^{2r_A} & 0 & 0 & 0 \\ 0 & e^{-2r_A} & 0 & 0 \\ 0 & 0 & e^{2r_B} & 0 \\ 0 & 0 & 0 & e^{-2r_B} \end{pmatrix}\begin{pmatrix} a+b & -c-d\varepsilon & a-b & -c+d\varepsilon \\ c+\frac{d}{\varepsilon} & a+b & c-\frac{d}{\varepsilon} & a-b \\ a-b & -c+d\varepsilon & a+b & -c-d\varepsilon \\ c-\frac{d}{\varepsilon} & a-b & c+\frac{d}{\varepsilon} & a+b \end{pmatrix} \\ &= \begin{pmatrix} L_{AA} & L_{AB} \\ L_{AB}^T & L_{BB} \end{pmatrix}. \end{aligned} \qquad (2.56)$$

The elements for the L_{AA} are

$$\begin{aligned}
V_{11} &= \frac{1}{8}\left(e^{2r_A}(a+b)^2 + e^{-2r_A}\left(c+\frac{d}{\varepsilon}\right)^2 + e^{2r_B}(a-b)^2 + e^{-2r_B}\left(c-\frac{d}{\varepsilon}\right)\right) \\
V_{12} &= \frac{1}{8}\left(e^{2r_A}(a+b)(-c-d\varepsilon) + e^{-2r_A}(a+b)\left(c+\frac{d}{\varepsilon}\right) + e^{2r_B}(a-b)(-c+d\varepsilon) + e^{2r_B}(a-b)\left(c-\frac{d}{\varepsilon}\right)\right) \\
V_{21} &= \frac{1}{8}\left(e^{2r_A}(a+b)(-c-d\varepsilon) + e^{-2r_A}(a+b)\left(c+\frac{d}{\varepsilon}\right) + e^{2r_B}(a-b)(-c+d\varepsilon) + e^{2r_B}(a-b)\left(c-\frac{d}{\varepsilon}\right)\right) \\
V_{22} &= \frac{1}{8}\left(e^{2r_A}(-c-d\varepsilon)^2 + e^{-2r_A}(a+b)^2 + e^{2r_B}(-c+d\varepsilon)^2 + e^{-2r_B}(a-b)^2\right).
\end{aligned} \tag{2.57}$$

The elements for the L_{AB} are

$$\begin{aligned}
V_{13} &= \frac{1}{8}\left((e^{2r_A}+e^{2r_B})(a+b)(a-b) + (e^{-2r_A}+e^{-2r_B})\left(c+\frac{d}{\varepsilon}\right)\left(c-\frac{d}{\varepsilon}\right)\right) \\
V_{14} &= \frac{1}{8}\left(e^{2r_A}(a+b)(-c+d\varepsilon) + e^{-2r_A}(a-b)\left(c+\frac{d}{\varepsilon}\right) + e^{2r_A}(a-b)(-c-d\varepsilon) + e^{-2r_B}(a+b)\left(c-\frac{d}{\varepsilon}\right)\right) \\
V_{23} &= \frac{1}{8}\left(e^{2r_A}(a-b)(-c-d\varepsilon) + e^{-2r_A}(a+b)\left(c-\frac{d}{\varepsilon}\right) + e^{2r_B}(a+b)(-c+d\varepsilon) + e^{-2r_B}(a-b)\left(c+\frac{d}{\varepsilon}\right)\right) \\
V_{24} &= \frac{1}{8}\left((e^{2r_A}+e^{2r_B})(-c-d\varepsilon)(-c+d\varepsilon) + (e^{-2r_A}+e^{-2r_B})(a-b)(a+b)\right).
\end{aligned} \tag{2.58}$$

The elements for the L_{BB} are

$$\begin{aligned}
V_{33} &= \frac{1}{8}\left(e^{2r_A}(a-b)^2 + e^{-2r_A}\left(c-\frac{d}{\varepsilon}\right)^2 + e^{2r_B}(a+b)^2 + e^{-2r_B}\left(c+\frac{d}{\varepsilon}\right)\right) \\
V_{34} &= \frac{1}{8}\left(e^{2r_A}(a-b)(-c+d\varepsilon) + e^{-2r_A}(a-b)\left(c-\frac{d}{\varepsilon}\right) + e^{2r_B}(a+b)(-c-d\varepsilon) + e^{-2r_B}(a+b)\left(c+\frac{d}{\varepsilon}\right)\right) \\
V_{43} &= \frac{1}{8}\left(e^{2r_A}(a-b)(-c+d\varepsilon) + e^{-2r_A}(a-b)\left(c-\frac{d}{\varepsilon}\right) + e^{2r_B}(a+b)(-c-d\varepsilon) + e^{-2r_B}(a+b)\left(c+\frac{d}{\varepsilon}\right)\right) \\
V_{44} &= \frac{1}{8}\left(e^{2r_A}(-c+d\varepsilon)^2 + e^{-2r_A}(a-b)^2 + e^{2r_B}(-c-d\varepsilon)^2 + e^{-2r_B}(a+b)^2\right).
\end{aligned} \tag{2.59}$$

2.2.2 Analytical derivation of maximum entanglement

Similarly, the quantification of the entanglement dynamics for this system utilises logarithmic negativity,

$$E_{A:B}(t) = [0, -\ln(2\tilde{v}^-)] \tag{2.60}$$

$$\tilde{v}^- = 2^{-1/2}\sqrt{\sum(V_{sq}) - \sqrt{\sum(V_{sq})^2 - 4\det(V_{sq})}}, \tag{2.61}$$

where $\sum(V_{sq}) = \det L_{AA} + \det L_{BB} - 2\det L_{AB}$.

$$
\begin{aligned}
\det(V_{sq}) &= \frac{a^4b^4}{16} + \frac{a^2d^4}{16} + \frac{b^4d^4}{16} + \frac{c^4d^4}{16} + \frac{a^4b^2d^2}{8} + \frac{b^cc^4d^2}{8} + \frac{a^2b^4c^2}{8} + \frac{a^2c^2d^4}{8} + \frac{a^2b^2c^2d^2}{8} \\
&= \frac{\cosh^4(\varepsilon T)\cos^4 T + \cos^2 T\sinh^4(\varepsilon T) + \cosh^4(\varepsilon T)\sin^4(T) + sinh^4(\varepsilon T)\sin^4(\varepsilon T)}{16} \\
&\quad + \frac{\cosh^2(\varepsilon T)\sinh^2(\varepsilon T)(\cos^4 T + \sin^4 T)}{8} + \frac{\cos^2 T\sin^2 T(\cosh^4(\varepsilon T) + \sinh^4(\varepsilon T))}{8} \\
&\quad + \frac{\cosh^2(\varepsilon T)\sinh^2(\varepsilon T)\cos^2 T\sin^2 T}{4} \\
&= \frac{1}{16}
\end{aligned} \tag{2.62}
$$

$$
\begin{aligned}
\sum(V_{sq}) &= \frac{1}{16\varepsilon^2}\Big[2c^2d^2 + 8a^2b^2\varepsilon^2 + 4c^2d^2\varepsilon^2 + 2c^2d^2\varepsilon^4 + 4a^2d^2e^{-2(r_A+r_B)} + c^2d^2e^{-2(r_A-r_B)} \\
&\quad + c^2d^2e^{2(r_A-r_B)} + 4b^2c^2\varepsilon^2e^{-2(r_A+r_B)} + 4b^2c^2\varepsilon^2e^{2(r_A+r_B)} + 4a^2d^2\varepsilon^4e^{2(r_A+r_B)} \\
&\quad - 2c^2d^2\varepsilon^2e^{-2(r_A-r_B)} - 2c^2d^2\varepsilon^2e^{2(r_A-r_B)} + c^2d^2\varepsilon^4e^{-2(r_A-r_B)} + c^2d^2\varepsilon^4e^{2(r_A-r_B)} \\
&\quad + 8abcd\varepsilon + 8abcd\varepsilon^3 - 8abcd\varepsilon^3e^{2(r_A+r_B)} - 8abcd\varepsilon e^{-2(r_A+r_B)} \\
&= \frac{1}{16\varepsilon^2}\Big[2c^2d^2 + 8a^2b^2\varepsilon^2 + 4c^2d^2\varepsilon^2 + 2c^2d^2\varepsilon^4 + 8abcd\varepsilon + 8abcd\varepsilon^3 \\
&\quad + e^{-2(r_A+r_B)}(4a^2d^2 + 4b^2c^2\varepsilon^2 - 8abcd\varepsilon) \\
&\quad + e^{2(r_B-r_A)}(c^2d^2 - 2c^2d^2\varepsilon^2 + c^2d^2\varepsilon^4) \\
&\quad + e^{2(r_A+r_B)}(4b^2c^2\varepsilon^2 + 4a^2d^2\varepsilon^4 - 8abcd\varepsilon^3) \\
&\quad + e^{2(r_A-r_B)}(c^2d^2 - 2c^2d^2\varepsilon^2 + c^2d^2\varepsilon^4)\Big] \\
&= \frac{1}{64}\left[8\left(2ab + cd\left(\varepsilon + \frac{1}{\varepsilon}\right)\right)^2 + 16e^{-2(r_A+r_B)}\left(\frac{ad}{\varepsilon} - bc\right)^2 + 4c^2d^2e^{2(r_B-r_A)}\left(\varepsilon - \frac{1}{\varepsilon}\right)^2\right. \\
&\quad \left. + 16e^{2(r_A+r_B)}(bc - ad\varepsilon)^2 + 4c^2d^2e^{2(r_A-r_B)}\left(\varepsilon - \frac{1}{\varepsilon}\right)^2\right].
\end{aligned} \tag{2.63}
$$

Comparing with squeezing parameter r_A and r_B, η is sufficiently small for the following approximation:

$$
\varepsilon = \sqrt{1-2\eta} \approx 1. \tag{2.64}
$$

Hence, $\sum(V_{sq})$ simplifies to

$$
\begin{aligned}
\sum(V_{sq}) &\approx \frac{1}{64}\left[8(2ab + 2cd)^2 + 16e^{-2(r_A+r_B)}(ad - bc)^2 + 16e^{2(r_A+r_B)}(bc - ad)^2\right] \\
&\approx \frac{1}{64}\left[8(2ab + 2cd)^2 + 16(bc - ad)^2\left(e^{-2(r_A+r_B)} + e^{2(r_A+r_B)}\right)\right] \\
&= \frac{1}{64}\left[32\cos^2(T - \varepsilon T)) + 32(\sin^2(T - \varepsilon T)\cosh(2(r_A + r_B))\right] \\
&= \frac{1}{64}\left[32\cos^2(T - \varepsilon T) + 32\sin^2(T - \varepsilon T) + 32\sin^2(T - \varepsilon T)\cosh(2(r_A + r_B)) - 32\sin^2(T - \varepsilon T)\right]
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{64}\left[32 + 32\sin^2(T - \varepsilon T)(\cosh(2(r_A + r_B)) - 1)\right] \\
&= \frac{1}{2} + \frac{1}{2}\sin^2(T - \varepsilon T)(\cosh(2(r_A + r_B)) - 1) \qquad (2.65)
\end{aligned}$$

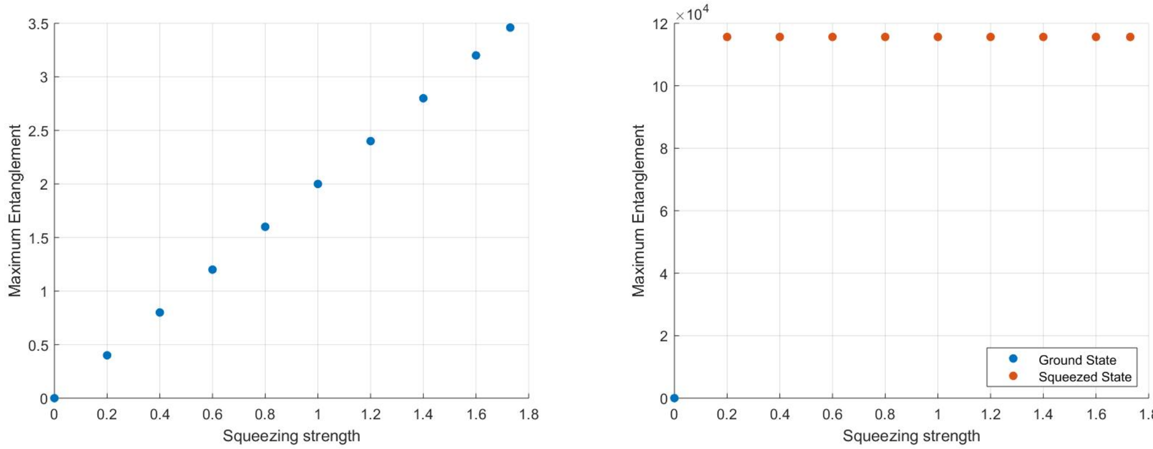
The maximal entanglement achievable in this setup can hence be derived. Again, in order to achieve maximum entanglement, $\sum(V)$ has to be maximised.
To achieve maximum $\sum(V)$, $\sin^2(T_{max} - \varepsilon T_{max}) = 1$,

$$\begin{aligned}
\sin^2(T_{max} - \varepsilon T_{max}) &= 1 \\
T_{max} &= \frac{\left(n + \frac{1}{2}\right)\pi}{1 - \varepsilon} \\
&\approx \frac{\left(n + \frac{1}{2}\right)\pi}{1 - \left(1 - \eta - \frac{\eta^2}{2}\right)} \\
&\approx \frac{\left(n + \frac{1}{2}\right)\pi}{\eta + \frac{\eta^2}{2}}, \quad n \in \mathbb{Z} \qquad (2.66)
\end{aligned}$$

$$\begin{aligned}
\sum(V_{sq})_{max} &= \frac{1}{2}[1 + \cosh(2(r_A + r_B)) - 1] \\
&= \frac{1}{2}\cosh(2(r_A + r_B)) \qquad (2.67) \\
\tilde{v}^-_{max} &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\cosh(2(r_A + r_B)) - \left[\frac{1}{4}\cosh^2(2(r_A + r_B)) - \frac{1}{4}\right]^{1/2}\right]^{1/2} \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\cosh(2(r_A + r_B)) - \frac{1}{2}(\sinh(2(r_A + r_B))\right]^{1/2} \\
&= \frac{1}{2}\left[e^{-2(r_A + r_B)}\right]^{1/2} \qquad (2.68) \\
E^{A:B}_{max} &= -\ln(2\tilde{v}^-_{max}) \\
&= -\ln\left[\left(e^{-2(r_A + r_B)}\right)^{1/2}\right] \\
&= |r_A + r_B|. \qquad (2.69)
\end{aligned}$$

Remarkably, the resulting maximum entanglement achievable by two squeezed ground state is only dependent on their relative squeezing strengths.

2.2.3 Computational analysis



(a) Evolution of maximally acheivable entanglement with varying squeezing factor

(b) Time required to reach maximal entanglement with different squeezing strength

Figure 2.3: Entanglement (Logarithmic negativity) dynamics for different squeezing strength

The main concern for squeezed state is the overlapping of the wave functions of the harmonic oscillators due to the small initial distance of separation. Recent experiment [25] has successfully squeezed light to approximately 15dB, corresponding to squeezing strength of $r = 1.73$, hence using it as an upper limit, the position uncertainty of the squeezed harmonic oscillator is approximated as

$$\begin{aligned} \Delta x &\approx \sqrt{\frac{\hbar}{2m\omega}e^{r}} \\ &= \sqrt{\frac{\hbar}{0.2}e^{1.73}} \approx 1.29 \times 10^{-16}\text{m}, \end{aligned} \tag{2.70}$$

which is 14 orders of magnitude smaller than the initial distance of separation for the spheres.

The maximum achievable entanglement scales linearly with the squeezing strength on the harmonic oscillators. These values of entanglement agree with the analytical derivation calculated in Eq (2.100), $E^{A:B}_{max} = |r_A + r_B|$. With a squeezing strength of 1.73, the calculated logarithmic negativity is 3.46, which far exceeds the minimum detectable entanglement with current optomechanical setup.

While squeezing greatly increases the entanglement, the time scale required for the system to reach maximum entanglement is significantly increased simultaneously. The disadvantage of the large time scale arises from the possibility of quantum decoherence of the entangled system due to environment. For our case, we consider the decoherence effect from the interaction of thermal photons and air molecules as the other sources are negligible. The interactions of the thermal photons limit the coherence time of the osmium oscillators to approximately 10^3s. By encapsulating the experiment in a vacuum chamber with 10^{10} molecules per m^3, the coherence time can be increased to approximately 10^5s [26]. The limits of these coherence time restrict the systems from evolving to achieve maximum entanglement. However, by taking the current detectable entanglement of $E \approx 0.01$, our experimental setup requires

$$t_{E_{A:B}(t)\approx 0.01} = 42.1\text{s}, \tag{2.71}$$

which is well within the limits of coherence time.

2.3 Thermal state

We now introduce environment to the systems via thermal interactions. The effect of thermal environment on the entanglement of the bipartite system can be performed by initialising the quantum harmonic oscillators in thermal states. This creates an energy distribution on the quantum harmonic oscillators whereby the probability of the oscillators to be observed in a certain energy state is governed by Bose-Einstein distribution.

2.3.1 Derivation of covariance matrix

Similar to the case of squeezed ground state, prior to the calculation of the covariance matrix for thermal state, we first define the respective expectation values of operators using density matrix of thermal states defined in Eq (1.37).

$$\langle \bar{x} \rangle = 0 \tag{2.72}$$

$$\langle \bar{p} \rangle = 0 \tag{2.73}$$

$$\langle \bar{x}^2 \rangle = \frac{1}{2} + N \tag{2.74}$$

$$\langle \bar{p}^2 \rangle = \frac{1}{2} + N \tag{2.75}$$

$$\langle \bar{p}\bar{x} + \bar{x}\bar{p} \rangle = 0. \tag{2.76}$$

The covariance matrix at $\omega t = 0$ for thermal statesis derived to be:

$$V_{th}(0) = \begin{pmatrix} \frac{1}{2}+N_A & 0 & 0 & 0 \\ 0 & \frac{1}{2}+N_A & 0 & 0 \\ 0 & 0 & \frac{1}{2}+N_B & 0 \\ 0 & 0 & 0 & \frac{1}{2}+N_B \end{pmatrix}, \tag{2.77}$$

where $N_A = \frac{1}{e^{\hbar\omega/kT_A}-1}$ and $N_B = \frac{1}{e^{\hbar\omega/kT_B}-1}$ for temperature T_A and T_B of the oscillators respectively. Full derivations can be found in Appendix 5.3.3.

2.3.2 Analytical derivation of maximum entanglement

By subjecting both of the oscillators to identical thermal environment ($T_A = T_B$), the covariance matrix of the thermal state is simplified

$$\begin{aligned} V_{th}(0) &= \begin{pmatrix} \frac{1}{2}+N & 0 & 0 & 0 \\ 0 & \frac{1}{2}+N & 0 & 0 \\ 0 & 0 & \frac{1}{2}+N & 0 \\ 0 & 0 & 0 & \frac{1}{2}+N \end{pmatrix} \\ &= (2N+1)V_{gd}(0), \end{aligned} \tag{2.78}$$

which is just the covariance matrix of the ground state harmonic oscillator scaled by a factor of $(2N+1)$. This expedites the process of calculating the analytical forms for the maximal achievable entanglement

$$\begin{aligned} \det(V_{th}) &= \det((2N+1)V_{gd}) \\ &= \left[\frac{1}{2}(2N+1)^2\right]^4 \end{aligned} \tag{2.79}$$

$$\begin{aligned} \sum(V_{th}) &= \sum((2N+1)V_{gd}) \\ &\approx \frac{1}{2}\left[1+2\eta^2\sin^2\left(\left(1-\eta-\frac{\eta^2}{2}\right)T\right)\right](2N+1)^2. \end{aligned} \tag{2.80}$$

Similarly, to achieve maximum entanglement, we maximise the value of $\sum(V_{th})$ by equating $\sin^2\left(\left(1-\eta-\frac{\eta^2}{2}\right)T\right) = 1$

$$\begin{aligned} \tilde{v}^-_{max} &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(1+2\eta^2)(2N+1)^2 - \left[\left(\frac{1}{2}(1+2\eta^2)(2N+1)^2\right)^2 - \frac{1}{4}(2N+1)^8\right]^{1/2}\right]^{1/2} \\ &\approx \frac{1}{\sqrt{2}}(2N+1)\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]^{1/2} \end{aligned} \tag{2.81}$$

$$\begin{aligned} E^{A:B}_{max} &= -\ln(2\tilde{v}^-_{max}) \\ &\approx -\ln\left[2\left\{\frac{1}{2}\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]\right\}^{1/2}(2N+1)\right] \\ &= -\ln\left[4\left\{\frac{1}{2}\left[\frac{1}{2}+\eta^2-2\eta\left(1+\frac{1}{2}\eta^2\right)\right]\right\}\right]^{1/2} - \ln(2N+1) \\ &\approx \eta - \ln(2N+1). \end{aligned} \tag{2.82}$$

With the introduction of thermal states, the extent of maximum entanglement is greatly reduced by a factor of $\ln(2N+1)$. Already in the regime of picokelvin, $\ln(2N+1) >> \eta$, resulting in negligible entanglement generated. As such, experimental setup to detect observable entanglement should be initialised very close to the ground states.

2.4 Squeezed thermal state

In the previous section, we have seen that thermal ground state practically does not provide any observable entanglement. We shall now explore the possibility of squeezed thermal states. The squeezed thermal state provides us with a fairly realistic experimental setup whereby the thermal state of the oscillators are subjected to squeezing operations.

2.4.1 Derivation of covariance matrix

The density matrix of squeezed thermal state is initialised as $\rho_{tsq} = \hat{S}\rho_{th}\hat{S}^{\dagger}$.
The procedure for deriving squeezed thermal states is identical to the derivation of squeezed ground state, with the exception of replacing ground state density matrix with thermal state density matrix. The expectation values of operators for squeezed thermal states are

$$\langle \bar{x} \rangle = 0 \tag{2.83}$$

$$\langle \bar{p} \rangle = 0 \tag{2.84}$$

$$\langle \bar{x}^2 \rangle = \frac{2N+1}{2}(\cosh(2r) - \sinh(2r)\cos(\phi)) \tag{2.85}$$

$$\langle \bar{p}^2 \rangle = \frac{2N+1}{2}(\cosh(2r) + \sinh(2r)\cos(\phi)) \tag{2.86}$$

$$\langle \bar{x}\bar{p} + \bar{p}\bar{x} \rangle = -(2N+1)\cosh(r)\sinh(r)\sin(\phi). \tag{2.87}$$

By using N_A, N_B for different thermal state subsystems and $(r_A, \phi_A), (r_B, \phi_B)$ of different parameters on the respective thermal states, we can derive the terms for the covariance matrix. For intermodal correlation, the single mode operators act on different subsystem, resulting in the reduction of those terms to zero. This can be observed in the derivation of the covariance matrix for thermal state. Hence, the resulting non-zero terms are local mode correlation for the respective subsystems.
Similarly, by applying the squeezing function on both the thermal states in the momentum space (i.e. $\phi_A = \phi_B = \pi$), the resulting covariance matrix is:

$$V_{tsq}(0) = \begin{pmatrix} \left(\frac{1}{2}+N_A\right)e^{2r_A} & 0 & 0 & 0 \\ 0 & \left(\frac{1}{2}+N_A\right)e^{-2r_A} & 0 & 0 \\ 0 & 0 & \left(\frac{1}{2}+N_B\right)e^{2r_B} & 0 \\ 0 & 0 & 0 & \left(\frac{1}{2}+N_B\right)e^{-2r_B} \end{pmatrix}. \tag{2.88}$$

Full derivations can be found in Appendix 5.3.4

2.4.2 Analytical derivation of maximum entanglement

Similar to the case of thermal ground state, assume identical temperature for both oscillators, the covariance matrix for squeezed thermal state can be re-written as a scaled covariance matrix of squeezed ground states

$$\begin{aligned} V_{tsq}(0) &= (2N+1)\begin{pmatrix} \frac{e^{2r_A}}{2} & 0 & 0 & 0 \\ 0 & \frac{e^{-2r_A}}{2} & 0 & 0 \\ 0 & 0 & \frac{e^{2r_B}}{2} & 0 \\ 0 & 0 & 0 & \frac{e^{-2r_B}}{2} \end{pmatrix} \\ &= (2N+1)V_{sq}(0), \end{aligned} \tag{2.89}$$

and the calculation for the sympletic matrix is an extension from the calculation of squeezed ground states.

$$\begin{aligned} \det(V_{tsq}) &= \det((2N+1)V_{sq}) \\ &= \left[\frac{1}{2}(2N+1)^2\right]^4 \end{aligned} \tag{2.90}$$

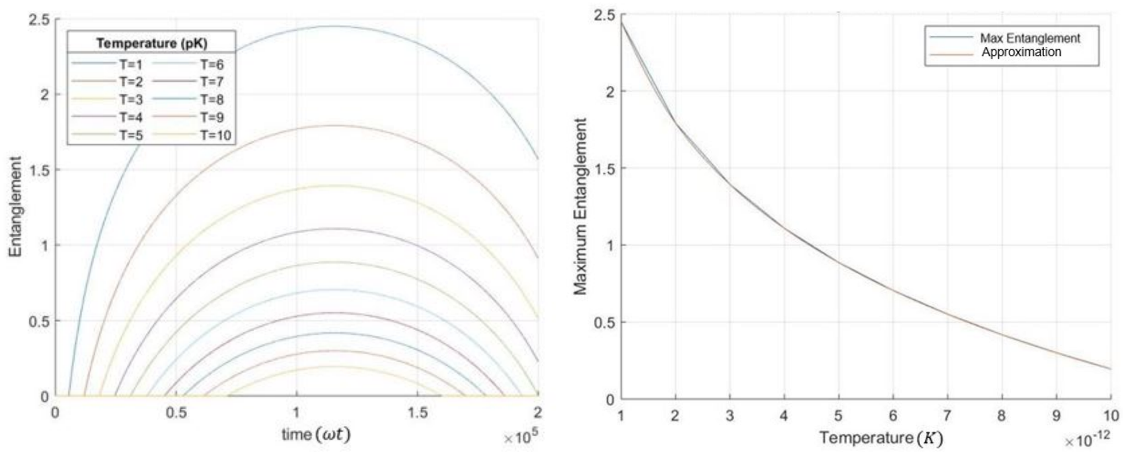
$$\begin{aligned} \sum(V_{tsq}) &= \sum((2N+1)V_{sq}) \\ &\approx \frac{1}{2} + \frac{1}{2}\sin^2(T-\varepsilon T)(\cosh(2(r_A+r_B))(2N+1)^2-1). \end{aligned} \tag{2.91}$$

Equating $\sin^2(T-\varepsilon T)=1$, the maximum entanglement accumulation is calculated as:

$$\begin{aligned} \tilde{v}^-_{max} &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\cosh(2(r_A+r_B))(2N+1)^2 - \left[\frac{1}{4}\cosh^2(2(r_A+r_B))(2N+1)^4 - \frac{1}{4}(2N+1)^8\right]^{1/2}\right]^{1/2} \\ &\approx \frac{1}{2}(2N+1)\left[e^{-2(r_A+r_B)}\right]^{1/2} \end{aligned} \tag{2.92}$$

$$\begin{aligned} E_{max} &= -\ln(2\tilde{v}^-_{max}) \\ &= -\ln\left[(2N+1)\left(e^{-2(r_A+r_B)}\right)^{1/2}\right] \\ &= |r_A+r_B| - \ln(2N+1). \end{aligned} \tag{2.93}$$

2.4.3 Computational analysis



(a) Evolution of entanglement starting with squeezed thermal state at variuos temperature

(b) Maximum achievable entanglement for different thermal environment. The data is fitted with approximated analytical derivation

Figure 2.4: Entanglement (Logarithmic negativity) dynamics for different thermal environment (varying T) starting with squeezed thermal state with squeezing strength $r_A = r_B = r = 1.73$

With squeezed thermal state, we observe significant entanglement accumulation at pico-Kelvin regime. The time required to achieve detectable entanglement increases with increasing temperature as represented by an increasing initial zero-valued logarithmic negativity with increasing temperature as depicted in Figure (2.4a). The time required to reach maximum entanglement, however, remains constant albeit varying temperature, which agrees with the analytical calculation in the previous section. Best fitting of maximum entanglement is achieved against temperature with the function

$$E_{max} \quad = \quad 3.46 - \ln\left[\frac{2}{e^{\hbar\omega/kT}} - 1\right]. \tag{2.94}$$

This shows that appropriate approximations are made and the analytical derivation for the system is accurate. In order to optimise the experimental setup to detect entanglement, we investigate the time required to acheive logarithmic negativity of 0.01 for different temperature. This value should be much smaller than the limits of coherence time due to thermal photons and air molecules explained previously.

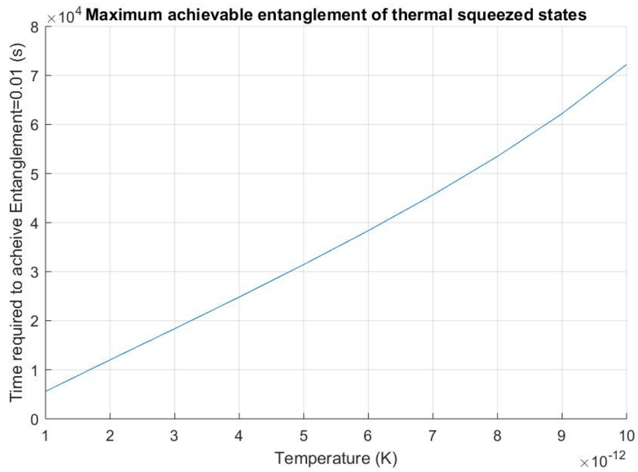


Figure 2.5: Time required to achieve logarithmic negativity of 0.01 for thermal states at different temperature

It is evidence that in the limit of pico-Kelvin, the quantum decoherence due to air molecules is negligible assuming the experiment is conducted in an environment of 10^{10} molecules per m^3. However, the effect of thermal photons is dramatic even for pico-Kelvin temperatures, the time required to achieve 0.01 logarithmic negativity is approximated to be more than 5×10^3s.

With the possible evaluations for different configurations of the experimental setup using quantum harmonic oscillators, we have shown the possibility of observing entanglement with certain parameters. However, these parameters are very challenging and difficult to incorporate with the current technologies. As such, we seek alternative experimental setup with more attainable parameters.

2.5 Casimir force

The parameter for initial separation of the centre of mass for the spheres are made under the consideration of maximising the gravitational attraction and minimising Casimir force. Casimir force arises from the second quantisation of electric fields. The energy of these quantised electric fields is expressed as

$$\langle E \rangle \quad = \quad \frac{1}{2} \sum_n E_n, \tag{2.95}$$

where E_n is the energy contribution from n^{th} mode standing wave. For objects which are close to one another, the separation restricts the maximal mode of the standing wave of electric waves trapped within. This results in a net Casimir energy, pushing the two objects towards each other. At near distances, the effect of Casimir's force can outweights the graviational effect, masking relevant results in experimental setup. However, we will show below that, with the parameters we consider, the Casimir energy is irrelevant to entanglement dynamics.

One technique to evaluate Casimir force between smooth objects at small distances is the Proximity Force

Approximation. In this case, we are interested in evaluating the Casimir's force between two smooth spheres. Analytical derivation to the Casimir energy of two smooth spheres at the limit of $R/L \approx 0.5$, where R is the radius of the spheres and L is the separation distance for their centre of masses, is derived to be [27]

$$\mathcal{E}_{PFA} = -\frac{\pi^3}{1440}\frac{\hbar c R}{(L-2R)^2}. \tag{2.96}$$

By setting the initial separation to $2.1R$, the parameters of our experimental setup are sufficiently close to the limit, and using $\frac{x_A - x_B}{L-2R} << 1$, we can approximate the Casimir's energy as

$$\begin{aligned}\mathcal{E}_{PFA} &= -\frac{\pi^3}{1440}\frac{\hbar c R}{(L-2R)^2} \\ &\approx -\frac{\pi^3}{1440}\frac{\hbar c R}{(L-2R)^2}\left[1+\frac{2(\hat{x}_A-\hat{x}_B)}{(L-2R)}+\frac{3(\hat{x}_A-\hat{x}_B)^2}{(L-2R)^2}\right].\end{aligned} \tag{2.97}$$

The approximation of the Casimir's energy to the second order terms are justified by the fact that second order terms are important in generating entanglement and hence we want to study its effect. Higher order terms are negligible as the standard deviation of the oscillators' position, Δx, is much smaller than the separation between the separation between the oscillators.

With the inclusion of Casimir's energy, the Hamiltonian of the system can be expressed as

$$\begin{aligned}\hat{H} &\approx \frac{\hat{p}_A^2}{2m}+\frac{1}{2}m\omega^2\hat{x}_A^2+\frac{\hat{p}_B^2}{2m}+\frac{1}{2}m\omega^2\hat{x}_B^2 \\ &\quad -\frac{Gm^2}{R}\left[1+\frac{(\hat{x}_A-\hat{x}_B)}{L}+\frac{(\hat{x}_A-\hat{x}_B)^2}{L^2}\right] \\ &\quad -\frac{\pi^3}{1440}\frac{\hbar c R}{(L-2R)^2}\left[1+\frac{2(\hat{x}_A-\hat{x}_B)}{(L-2R)}+\frac{3(\hat{x}_A-\hat{x}_B)^2}{(L-2R)^2}\right] \\ &= \frac{\hat{p}_A^2}{2m}+\frac{1}{2}m\omega^2\hat{x}_A^2+\frac{\hat{p}_B^2}{2m}+\frac{1}{2}m\omega^2\hat{x}_B^2 \\ &\quad -\alpha-\beta(\hat{x}_A-\hat{x}_B)-\gamma(\hat{x}_A-\hat{x}_B)^2,\end{aligned} \tag{2.98}$$

where $\alpha = \frac{Gm^2}{R} + \frac{\pi^3\hbar c R}{1440(L-2R)^2}$, $\beta = \frac{Gm^2}{R^2} + \frac{2\pi^3\hbar c R}{1440(L-2R)^3}$, $\gamma = \frac{Gm^2}{R^3} + \frac{3\pi^3\hbar c R}{1440(L-2R)^4}$.

The corrected equations of motions are

$$\dot{\bar{x}}_A = \omega\bar{p}_A \tag{2.99}$$

$$\dot{\bar{p}}_A = \left(\frac{2\gamma}{m\omega}-\omega\right)\bar{x}_A - \frac{2\gamma}{m\omega}\bar{x}_B + \sqrt{\frac{1}{\hbar m\omega}}\beta \tag{2.100}$$

$$\dot{\bar{x}}_B = \omega\bar{p}_B \tag{2.101}$$

$$\dot{\bar{p}}_A = -\frac{2\gamma}{m\omega}\bar{x}_A + \left(\frac{2\gamma}{m\omega}-\omega\right)\bar{x}_B - \sqrt{\frac{1}{\hbar m\omega}}\beta, \tag{2.102}$$

and the matrix form is

$$\begin{pmatrix} \dot{\bar{x}}_A \\ \dot{\bar{p}}_A \\ \dot{\bar{x}}_B \\ \dot{\bar{p}}_B \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \left(\frac{2\gamma}{m\omega} - \omega\right) & 0 & -\frac{2\gamma}{m\omega} & 0 \\ 0 & 0 & 0 & \omega \\ -\frac{2\gamma}{m\omega} & 0 & \left(\frac{2\gamma}{m\omega} - \omega\right) & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_A \\ \bar{p}_A \\ \bar{x}_B \\ \bar{p}_B \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{\frac{1}{\hbar m\omega}}\beta \\ 0 \\ -\sqrt{\frac{1}{\hbar m\omega}}\beta \end{pmatrix} \tag{2.103}$$

By allowing $\eta = \frac{2\gamma}{m\omega^2}$, the K-matrix is reduced to the initial form

$$K = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega(\eta - 1) & 0 & -\eta\omega & 0 \\ 0 & 0 & 0 & \omega \\ -\eta\omega & 0 & \omega(\eta - 1) & 0 \end{pmatrix}. \tag{2.104}$$

Evaluating the terms for η,

$$\begin{aligned} \frac{\text{Casimir's contribution}}{\text{Gravitational contribution}} &= \frac{\frac{3\pi^3\hbar c R}{1440(L-2R)^4}}{\frac{Gm^2}{R^3}} \\ &= \frac{3\pi^3\hbar c}{1440 G m^2 (0.1)^4} \approx 3.06 \times 10^{-13}. \end{aligned} \tag{2.105}$$

Hence, we conclude that the Casimir's effect is negligible compared to Gravitational effect for masses on the order of a kilogram. As such, η can be approximated to the original form, and the calculations of the entanglement dynamics remain unperturbed.

Chapter 3

Entanglement dynamics between free falling massive particles

We now consider two spheres initially trapped in a harmonic trap and cooled. At $t = 0$, the traps are removedand the spheres interact purely via gravitational interactions. Prior to the quantum treatment of the setup, we first consider classical mechanics, allowing us to calculate useful dynamics such as the time required for collision.

3.1 Classical framework

Consider two spheres of mass m_A and m_B, each with radius R. The two spheres are intially separated by distance L and trapped in harmonic trap of frequency ω. Upon releasing from the trap, the spheres interacts purely by Newtonian gravity.

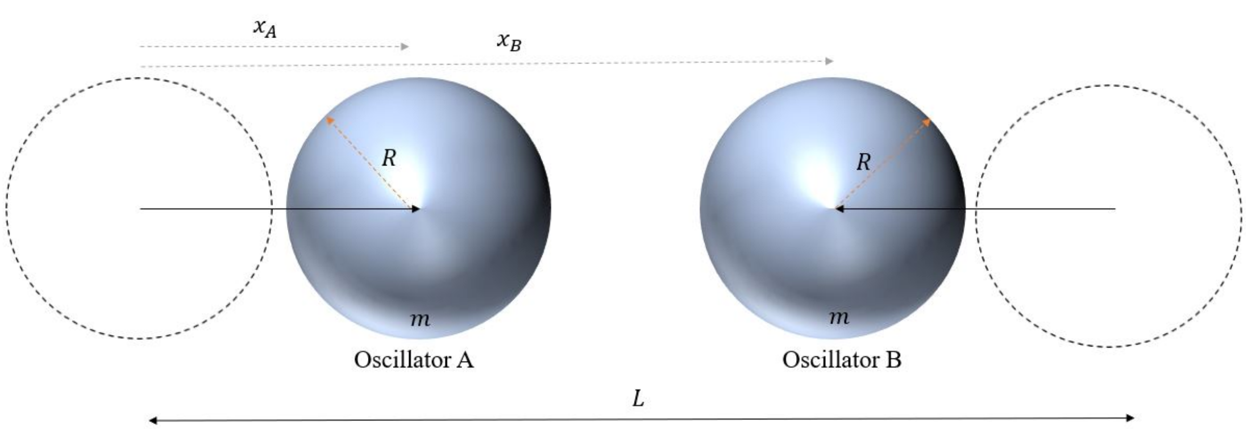


Figure 3.1: Classical spheres interacting purely via gravitational force get closer to one another

The dynamics of the masses can be expressed as:

$$
\begin{aligned}
F_{BA} &= \frac{Gm_A m_B}{(x_B - x_A)^2} = m_A \ddot{x}_A \\
F_{AB} &= -\frac{Gm_A m_B}{(x_B - x_A)^2} = m_B \ddot{x}_B \\
\ddot{x}A &= \frac{Gm_B}{(x_B - x_A)^2} && (3.1) \\
\ddot{x}_B &= -\frac{Gm_A}{(x_B - x_A)^2}. && (3.2)
\end{aligned}
$$

Subtracting Equation (2) from Equation (1) and mapping the variable $x_B - x_A = r$,

$$
\begin{aligned}
\ddot{x_B} - \ddot{x_A} &= \frac{d^2}{dt^2}(x_B - x_A) = -\frac{G(m_A + m_B)}{(x_B - x_A)^2} \\
\ddot{r} &= -\frac{G(m_A + m_B)}{r^2} \\
a &= \frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr} = v\frac{dv}{dr} = -\frac{G(m_A + m_B)}{r^2} \\
\Rightarrow vdv &= -\frac{G(m_A + m_B)dr}{r^2}. && (3.3)
\end{aligned}
$$

At initial configuration of the system, $v = 0$ when $r = L$. Hence integrating Equation (3),

$$
\begin{aligned}
\int_0^v v' dv &= \int_L^r -\frac{G(m_A + m_B)}{r^2} dr \\
\frac{1}{2}v^2 &= \frac{G(m_A + m_B)}{r} - \frac{G(m_A + m_B)}{L} \\
v &= \frac{dr}{dt} = -\sqrt{\frac{2GL(m_A + m_B) - 2Gr(m_A + m_B)}{rL}} \\
dt &= -\frac{\sqrt{Lr}dr}{\sqrt{2GL(m_A + m_B) - 2Gr(m_A + m_B)}} \\
&= -\sqrt{\frac{L}{2G(m_A + m_B)}}\sqrt{\frac{r}{L - r}}dr. && (3.4)
\end{aligned}
$$

3.1.1 Time of collision

Performing another change of variable $u = \sqrt{\frac{r}{L-r}}$,

$$
\begin{aligned}
r &= \frac{Lu^2}{1 + u^2} \\
dr &= \frac{2Lu}{(1 + u^2)^2}du,
\end{aligned}
$$

At $r = L$, $u = \infty$, at $r = 2R$, $u = \sqrt{\frac{2R}{L-2R}}$,

$$\begin{aligned}
t &= -\frac{2L^{3/2}}{\sqrt{2G(m_A+m_B)}}\int_{\infty}^{\sqrt{\frac{2R}{L-2R}}}\frac{u^2}{(1+u^2)^2}du \\
&= \frac{2L^{3/2}}{\sqrt{2G(m_A+m_B)}}\left[\frac{1}{2}\left(\tan^{-1}u-\frac{u}{1+u^2}\right)\right]_{\sqrt{\frac{2R}{L-2R}}}^{\infty} \\
&= \frac{2L^{3/2}}{\sqrt{2G(m_A+m_B)}}\left[\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\sqrt{\frac{2R}{L-2R}}+\frac{\sqrt{2R(L-2R)}}{L}\right)\right] \\
&= \frac{L^{3/2}}{\sqrt{Gm}}\left[\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\sqrt{\frac{2R}{L-2R}}+\frac{\sqrt{2R(L-2R)}}{L}\right)\right], \text{(for } m_A = m_B).
\end{aligned} \tag{3.5}$$

This calculation is critical in providing an upper bound to the time for the quantum framework.

3.2 Quantum framework

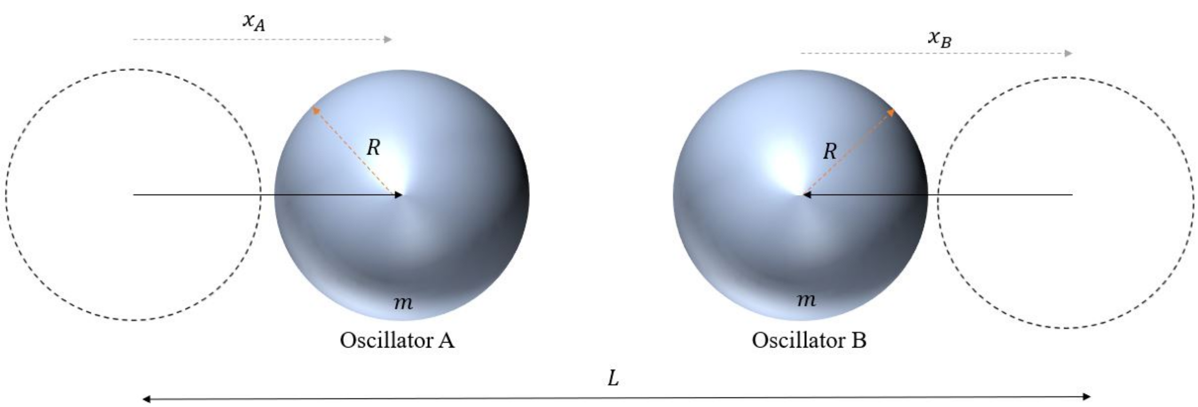


Figure 3.2: Two mechanical spheres trapped within individual harmonic traps. At $t = 0$, the spheres are released from the trap, and the spheres interact freely via gravitational force.

The quantum mechanical setup for the experiment is identical to the classical setup, with the exception of a new reference point for the position of oscillator B. The Hamiltonian of the system can be represented as

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}_A^2}{2m}+\frac{\hat{p}_B^2}{2m}-\frac{Gm^2}{(L-(\hat{x}_A-\hat{x}_B))} \\
&\approx \frac{\hat{p}_A^2}{2m}+\frac{\hat{p}_B^2}{2m}-\frac{Gm^2}{L}\left(1+\frac{\hat{x}_A-\hat{x}_B}{L}+\frac{(\hat{x}_A-\hat{x}_B)^2}{L^2}\right).
\end{aligned} \tag{3.6}$$

The justification to the second order approximation of the gravitational hamiltonian is identical to that of the case for quantum harmonic oscillators. However for the case of free particle, as the distance decreases, the Taylor approximation becomes increasingly inaccurate. As such, we only consider situations where $x_A - x_B < L/100$.

The equations of motions in Heisenberg picture can be expressed as:

$$
\begin{aligned}
\dot{\bar{x}}_A &= \omega \bar{p}_A \\
\dot{\bar{p}}_A &= \frac{2Gm}{\omega L^3}\bar{x}_A - \frac{2Gm}{\omega L^3}\bar{x}_B + \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}} \\
\dot{\bar{x}}_B &= \omega \bar{p}_B \\
\dot{\bar{p}}_B &= -\frac{2Gm}{\omega L^3}\bar{x}_A + \frac{2Gm}{\omega L^3}\bar{x}_B - \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}},
\end{aligned}
\tag{3.7}
$$

where ω is the oscillation frequency of the initial harmonic trap. We set up to calculate the time dependent covariance matrix using similar methodology as the quantum harmonic oscillator setup. Re-writting the Heisenberg equation in matrix form $\dot{u}(t) = Ku(t) + c$,

$$
\begin{pmatrix} \dot{\bar{x}}_A \\ \dot{\bar{p}}_A \\ \dot{\bar{x}}_B \\ \dot{\bar{p}}_B \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \frac{2Gm}{\omega L^3} & 0 & -\frac{2Gm}{\omega L^3} & 0 \\ 0 & 0 & 0 & \omega \\ -\frac{2Gm}{\omega L^3} & 0 & \frac{2Gm}{\omega L^3} & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_A \\ \bar{p}_A \\ \bar{x}_B \\ \bar{p}_B \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}} \\ 0 \\ \frac{Gm^2}{L^2}\sqrt{\frac{1}{\hbar m\omega}} \end{pmatrix}, \tag{3.8}
$$

we can redefine the K-matrix by making the substitution $\chi = \frac{2Gm}{\omega L^3}$

$$
K = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \chi & 0 & -\chi & 0 \\ 0 & 0 & 0 & \omega \\ -\chi & 0 & \chi & 0 \end{pmatrix}. \tag{3.9}
$$

In order to calculate the exponential matrix e^{Kt}, we implemented Sylvester's formula (Appendix 5.2.2) as the fundamental matrix solution does not apply to the non-invertible K-matrix. Solving for the eigenvalues

$$
\begin{aligned}
\det \begin{pmatrix} -\lambda & \omega & 0 & 0 \\ \chi & -\lambda & -\chi & 0 \\ 0 & 0 & -\lambda & \omega \\ -\chi & 0 & \chi & -\lambda \end{pmatrix} &= 0 \\
\lambda^4 - 2\chi\omega\lambda^2 &= 0 \\
\lambda_{1,2} = 0 \quad \text{or} \quad \lambda_{3,4} &= \pm\sqrt{2\chi\omega}.
\end{aligned}
\tag{3.10}
$$

As the eigenvalue $\lambda_{1,2} = 0$ has a degeneracy of 2, the matrix exponential can be expressed as the following

$$\begin{aligned} e^{Kt} &= C_1 e^{0t} + C_2 t e^{0t} + C_3 e^{\sqrt{2\chi\omega}t} + C_4 e^{-\sqrt{2\chi\omega}t} \\ &= C_1 + C_2 t + C_3 e^{\sqrt{2\chi\omega}t} + C_4 e^{-\sqrt{2\chi\omega}t}. \end{aligned} \tag{3.11}$$

In order to solve for the constant terms C_1, C_2, C_3, C_4, we differentiate Equation (3.11) to obtain the 3 other simultaneous equations

$$Ke^{Kt} = C_2 + \sqrt{2\chi\omega}C_3 e^{\sqrt{2\chi\omega}t} - C_4\sqrt{2\chi\omega}e^{-\sqrt{2\chi\omega}t} \tag{3.12}$$

$$K^2 e^{Kt} = 2\chi\omega C_3 e^{\sqrt{2\chi\omega}t} + 2\chi\omega C_4 e^{-\sqrt{2\chi\omega}t} \tag{3.13}$$

$$K^3 e^{Kt} = (2\chi\omega)^{3/2} C_3 e^{\sqrt{2\chi\omega}t} - (2\chi\omega)^{3/2} C_4 e^{-\sqrt{2\chi\omega}t}. \tag{3.14}$$

Equating $t = 0$ and solving for the constant matrices:

$$C_1 = \mathbb{1} - \frac{K^2}{2\chi\omega} \tag{3.15}$$

$$C_2 = K - \frac{K^3}{2\chi\omega} \tag{3.16}$$

$$C_3 = \frac{1}{2}\left[\frac{K^2}{2\chi\omega} + \frac{A^3}{(2\chi\omega)^{3/2}}\right] \tag{3.17}$$

$$C_4 = \frac{1}{2}\left[\frac{K^2}{2\chi\omega} - \frac{A^3}{(2\chi\omega)^{3/2}}\right], \tag{3.18}$$

the matrix exponential is evaluated as

$$e^{Kt} = \frac{1}{2}\begin{pmatrix} 1+\cosh(\sqrt{2\chi\omega}t) & \omega t + \sqrt{\frac{\omega}{2\chi}}\sinh(\sqrt{2\chi\omega}t) & 1-\cosh(\sqrt{2\chi\omega}t) & \omega t - \sqrt{\frac{\omega}{2\chi}}\sinh(\sqrt{2\chi\omega}t) \\ \sqrt{\frac{\chi}{2\omega}}\sinh(\sqrt{2\chi\omega}t) & 1+\cosh(\sqrt{2\chi\omega}t) & -\sqrt{\frac{\chi}{2\omega}}\sinh(\sqrt{2\chi\omega}t) & 1-\cosh(\sqrt{2\chi\omega}t) \\ 1-\cosh(\sqrt{2\chi\omega}t) & \omega - \sqrt{\frac{\omega}{2\chi}}\sinh(\sqrt{2\chi\omega}t) & 1+\cosh(\sqrt{2\chi\omega}t) & \omega t + \sqrt{\frac{\omega}{2\chi}}\sinh(\sqrt{2\chi\omega}t) \\ -\sqrt{\frac{\chi}{2\omega}}\sinh(\sqrt{2\chi\omega}t) & 1-\cosh(\sqrt{2\chi\omega}t) & \sqrt{\frac{\chi}{2\omega}}\sinh(\sqrt{2\chi\omega}t) & 1+\cosh(\sqrt{2\chi\omega}t) \end{pmatrix}. \tag{3.19}$$

The matrix representing local mode correlation and intermodal correlations are hence:
For subsystem A, L_{AA}

$$V_{11} = \frac{2\chi + 2\chi\cosh^2(\sqrt{2\chi\omega}t) + \omega\sinh^2(\sqrt{2\chi\omega}t) + 2\chi\omega^2 t^2}{8\chi} \tag{3.20}$$

$$V_{12} = \frac{2(\chi\omega)^{3/2}t + \sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \tag{3.21}$$

$$V_{21} = \frac{2(\chi\omega)^{3/2}t + \sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \tag{3.22}$$

$$V_{22} = \frac{3\omega - 2\chi + 2\chi\cosh(2\sqrt{2\chi\omega}t) + \omega\cosh(2\sqrt{2\chi\omega}t)}{8\omega}. \quad (3.23)$$

For subsystem B, L_{BB},

$$V_{11} = \frac{2\chi + 2\chi\cosh^2(\sqrt{2\chi\omega}t) + \omega\sinh^2(\sqrt{2\chi\omega}t + 2\chi\omega^2 t^2}{8\chi} \quad (3.24)$$

$$V_{12} = \frac{2(\chi\omega)^{3/2}t + \sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \quad (3.25)$$

$$V_{21} = \frac{2(\chi\omega)^{3/2}t + \sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \quad (3.26)$$

$$V_{22} = \frac{3\omega - 2\chi + 2\chi\cosh(2\sqrt{2\chi\omega}t) + \omega\cosh(2\sqrt{2\chi\omega}t)}{8\omega}. \quad (3.27)$$

For intermodal correlation, L_{AB},

$$V_{13} = -\frac{2\chi\sinh^2(\sqrt{2\chi\omega}t) + \omega\sinh^2(\sqrt{2\chi\omega}t) - 2\chi\omega^2 t^2}{8\chi} \quad (3.28)$$

$$V_{14} = \frac{\sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) - 2(\chi\omega)^{3/2}t + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \quad (3.29)$$

$$V_{23} = \frac{\sqrt{2}\chi^2\sinh(2\sqrt{2\chi\omega}t) - 2(\chi\omega)^{3/2}t + 2^{-1/2}\chi\omega\sinh(2\sqrt{2\chi\omega}t)}{8\sqrt{\chi\omega}\chi} \quad (3.30)$$

$$V_{24} = \frac{(2\chi + \omega)(2\sinh^2(\sqrt{2\chi\omega}t)}{8\omega}. \quad (3.31)$$

Similarly, we use logarithmic negativity as the quantifier for entanglement,

$$det(V_{fp}) = \frac{1}{16} \quad (3.32)$$

$$\begin{aligned}\sum(V_{fp}) &= -\frac{1}{8\chi^2\omega}[\chi\omega^2 - 4\chi^3\cosh^2(\sqrt{2\chi\omega}t) + 4\chi^3 - \chi\omega^2\cosh^2(\sqrt{2\chi\omega}t) - 4\chi^2\omega\cosh^2(\sqrt{2\chi\omega}t) \\ &+4\chi^3t^2\omega^2 - 2\chi^2t^2\omega^3\cosh^2(\sqrt{2\chi\omega}t) - 4\chi^3t^2\omega^2\cosh^2(\sqrt{2\chi\omega}t) \\ &+4\sqrt{2}\chi t(\chi\omega)^{3/2}\cosh(\sqrt{2\chi\omega}t)\sinh(\sqrt{2\chi\omega}t) + 2\sqrt{2}\omega t(\chi\omega)^{3/2}\cosh(\sqrt{2\chi\omega}t)\sinh(\sqrt{2\chi\omega}t)] \\ &= -\frac{1}{8\chi^2\omega}[2\sqrt{2}(\chi\omega)^{3/2}\cosh(\sqrt{2\chi\omega}t)\sinh(\sqrt{2\chi\omega}t)(2\chi t + \omega) \\ &-\sinh^2(\sqrt{2\chi\omega}t)(\chi\omega^2 + 4\chi^3 + 4\chi^3t^2\omega^2) - \cosh^2(\sqrt{2\chi\omega}t)(4\chi^2\omega + 2\chi^2t^2\omega^2)]. \quad (3.33)\end{aligned}$$

Taylor expanding sinh and cosh order up to the third terms (excluding constant), $\sinh(\sqrt{2\chi\omega}t) \approx \sqrt{2\chi\omega}t + \frac{(\sqrt{2\chi\omega}t)^3}{3!} + \frac{(\sqrt{2\chi\omega}t)^5}{5!}$ and $\cosh(\sqrt{2\chi\omega}t) \approx 1 + \frac{(\sqrt{2\chi\omega}t)^2}{2!} + \frac{(\sqrt{2\chi\omega}t)^4}{4!} + \frac{(\sqrt{2\chi\omega}t)^6}{6!}$, the above equation is simplified to

$$\begin{aligned}\sum(V_{fp}) \approx\ & \frac{1}{2} + \chi^2t^2 + \frac{\chi^2\omega^2t^4}{3} + \frac{\chi^2\omega^4t^6}{9} + \frac{2\chi^3\omega t^4}{3} + \frac{14\chi^3\omega^3t^6}{45} + \frac{13\chi^3\omega^5t^8}{288} + \frac{8\chi^4\omega^2t^6}{45} \\ &+ \frac{37\chi^4\omega^4t^8}{360} + \frac{\chi^4\omega^6t^{10}}{120} + \frac{\chi^5\omega^3t^8}{40} + \frac{19\chi^5\omega^5t^{10}}{1080} + \frac{49\chi^5\omega^7t^{12}}{64800} + \frac{\chi^6\omega^4t^{10}}{540} + \frac{\chi^6\omega^6t^{12}}{648}\end{aligned}$$

$$+\frac{\chi^6\omega^8 t^{14}}{32400}+\frac{\chi^7\omega^5 t^{12}}{16200}+\frac{\chi^7\omega^7 t^{14}}{16200}$$

$$\approx \frac{1}{2}+\frac{\chi^2\omega^4 t^6}{9}. \tag{3.34}$$

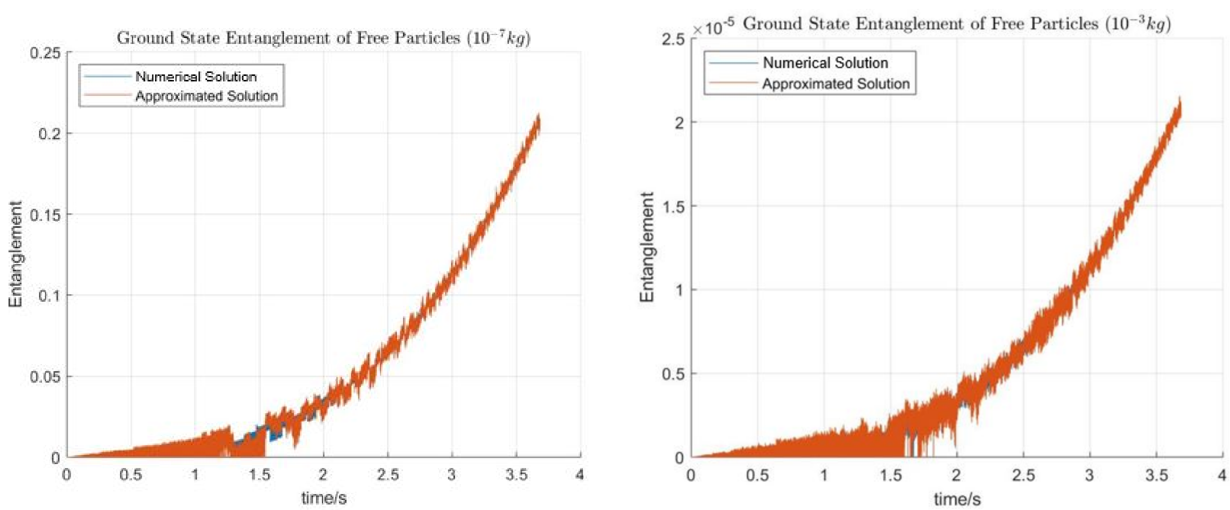
Hence, the negativity and logarithmic negativity dynamics of free interacting spheres are

$$\tilde{v}^- = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2}+\frac{\chi^2\omega^4 t^6}{9}\right)-\left[\left(\frac{1}{2}+\frac{\chi^2\omega^4 t^6}{9}\right)^2-\frac{1}{4}\right]^{1/2}\right]^{1/2} \tag{3.35}$$

$$\begin{aligned} E_{A:B}(t) &= -\ln(2\tilde{v}^-) \\ &= -\ln\left\{\sqrt{2}\left[\left(\frac{1}{2}+\frac{\chi^2\omega^4 t^6}{9}\right)-\left[\left(\frac{1}{2}+\frac{\chi^2\omega^4 t^6}{9}\right)^2-\frac{1}{4}\right]^{1/2}\right]^{1/2}\right\} \\ &= -\frac{1}{2}\ln\left[\left(1+\frac{\chi^2\omega^4 t^6}{9}\right)-\left[\left(1+\frac{\chi^2\omega^4 t^6}{9}\right)^2-1\right]^{1/2}\right]. \end{aligned} \tag{3.36}$$

3.3 Computational analysis

For free particles, we are interested in the effect of various masses on the entanglement accumulation. From [28], we obtain the current experimental limits for the trapping of mechanical oscillators in harmonic traps of frequency ω. The relationship between the mass of the oscillators and the trapping frequency is given as $m\omega \sim 10^{-1}$.



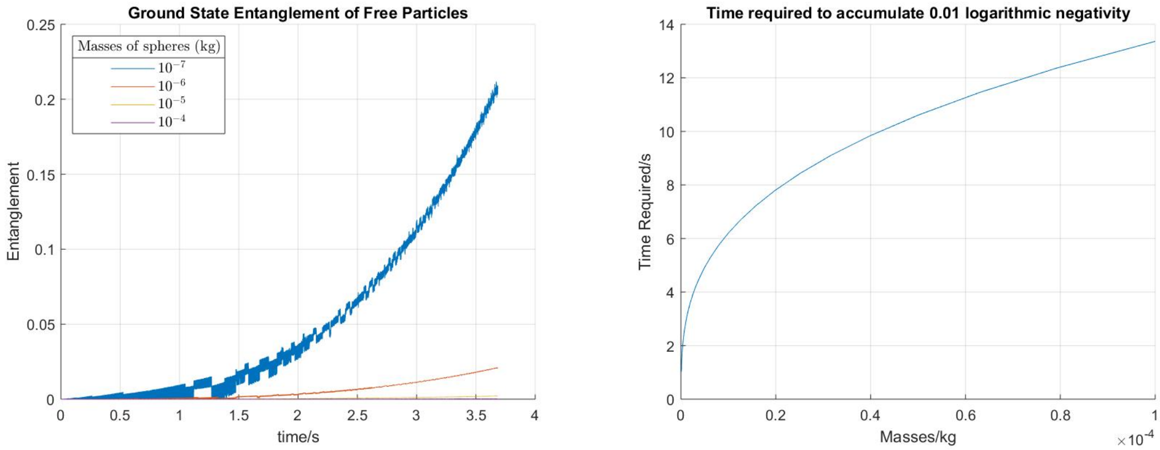
(a) Comparison of approximated and exact solution for entanglement dynamics of freely interacting spheres with parameters: $m = 10^{-7}$kg, $\omega = 10^6$Hz, $L = 10$R.

(b) Comparison of approximated and exact solution for entanglement dynamics of freely interacting spheres with parameters: $m = 10^{-3}$kg, $\omega = 10^2$Hz, $L = 10$R.

Figure 3.3: Evaluations of approximate results for entanglement (Logarithmic negativity) dynamics of freely interacting spheres from time $t = 0$s to $t = 3.5$s

A comparison between the solution calculated in matrix form and approximated solution calculated in Eq

(3.36) showed closed correlations. While certain fluctuations within the plots are not fully characterised by the approximated solution, it can still be used to evaluate the general entanglement dynamics.



(a) (b)

Figure 3.4: Entanglement dynamics for different masses, ranging from $m = 10^{-4}$kg to $m = 10^{-7}$kg

The entanglement dynamics of freely interacting particles shows promising result. For 100 ng spheres, the logarithmic negativity of 0.01 is accumulated in approximately 1 second. The result of short accumulation time implies that this experimental setup is less susceptible to the decoherence. Another advantage of this setup is the initial separation distance between the spheres. In this case, we set the initial separation for the centre of masses of both spheres to be 10 times its radius. This is a much more achievable setup with the current technology compared to the previous $L = 2.1R$. From the time required to reach 0.01 logarithmic negativity as a function of mass, we observe that for the case of freely interacting spheres, the small spheres have advantage over larger spheres. We hypothesise that with smaller masses, the spreading of the wavefunctions for individual spheres occurs at a faster rate, resulting in faster accumulations of entanglement due to close proximity interactions. This provides yet another advantage of this setup over oscillators as smaller spheres are easier to cool down to ground states and trapping them is easier with lesser repurcussions from noises.

Chapter 4

Conclusion and future works

Analytical and computational analysis of the gravitational entanglement between oscillators and free falling spheres has shown the feasibility of gravity mediated entanglement. The purpose of our research is to probe the quantum properties of gravity as well as provide suitable figures of merit for the reference for future related experiments. For the case of oscillators, squeezing proved to be effective in increasing the entanglement between the subsystems. Introduction of thermal environment provided an upper bound to the temperature required to cool the system for observable entanglement with current technology. Pico-Kelvin temperature range is currently unachievable at such large scales with kilogram mirror setups. For the case of freely falling spheres, most of the parameters discussed are achievable with the current technology. The interation of optomechanical systems and free falling vacuum environment are optimal to probe the gravity mediated entanglement.

While we have considered several parameters in the analytical calculations, there were few that were left out for future considerations. In the calculation of oscillators, we assumed zero damping in the mirrors. The incorporation of damping factors provides a more realistic simulations of oscillators. For both the proposed setups, the introduction of noise in the form of Langevin treatment will be optimal in increasing the accuracy of our models. Finally, we discussed the entanglement gain between two smooth spheres, which are hard to manufacture for experimental purposes. We hope to be able to extend our calculations to arbitrary shapes of interacting objects.

Chapter 5

Appendix

5.1 Gravitational Hamiltonian of sphere

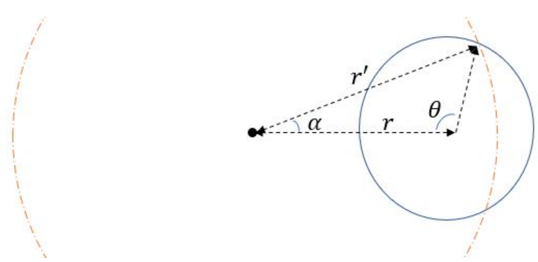


Figure 5.1: Energy of a point mass and a spherical mass system

Consider a system with a point mass and a sphere with both mass m. The Hamiltonian of the system can be calculated by integrating the effect of potential for the point mass on the sphere,

$$\begin{aligned}
U &= \int V(r')dm \\
&= \int \frac{Gm}{r'}\rho_m dV \quad \text{,where } \rho_m \text{ is the mass density} \\
&= \int \frac{Gm}{r'}\rho_m \int_{\phi=0}^{2\pi}\int_{\alpha=0}^{\alpha} r'^2 \sin\alpha d\phi d\alpha dr' \\
&= \int Gm\rho_m 2\pi r'(1-\cos\alpha)dr' \\
&= \int Gm\rho_m 2\pi(1-\cos\alpha)rR\sin\theta d\theta \quad \text{,where } r'^2 = r^2+R^2-2rR\cos\theta \\
&= 2\pi Gm\rho_m \int_0^{\pi}\left(1-\frac{r-R\cos\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}\right)rR\sin\theta d\theta \\
&= 2\pi Gm\rho_m \left[2rR - \int_0^{\pi}\left(\frac{r-R\cos\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}\right)rR\sin\theta d\theta\right]. \qquad (5.1)
\end{aligned}$$

In order to solve the integration, multiple substitution has to be made. We fist substitute $u = \cos\theta$,

$$\int_0^\pi \left(\frac{r - R\cos\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}\right) rR\sin\theta d\theta \quad = \quad -rR\int_1^{-1} \frac{r - Ru}{\sqrt{r^2 + R^2 - 2rRu}} du. \tag{5.2}$$

Next, we let $s = \sqrt{r^2 + R^2 - 2rRu}$,

$$\begin{aligned} -rR\int_1^{-1} \frac{r - Ru}{\sqrt{r^2 + R^2 - 2rRu}} &= -rR\int_{r-R}^{r+R} \frac{r - Ru}{\sqrt{r^2 + R^2 - 2rRu}} \frac{\sqrt{r^2 + R^2 - 2rRu}}{rR} ds \\ &= -\int_{r+R}^{r-R} r + \frac{1}{2r}(s^2 - r^2 - R^2) ds \\ &= 2rR + \frac{1}{2r}\frac{4}{3}R^3. \end{aligned} \tag{5.3}$$

Substituting back to the original equation,

$$\begin{aligned} U &= 2\pi Gm \frac{m}{\frac{4}{3}\pi R^3} \frac{1}{2r} \frac{4}{3} R^3 \\ &= \frac{Gm^2}{r}. \end{aligned} \tag{5.4}$$

Hence, we can conclude that the Hamiltonian between two point masses is the same as the Hamiltonian between two spheres.

5.2 Matrix Exponential

5.2.1 Fundamental Matrix Solution

Consider a first order differential equation for vector of n-dimension $\vec{x} = \begin{bmatrix} x_A(t) & x_B(t) & \cdots & x_n(t) \end{bmatrix}^T$,

$$\frac{d\vec{x}}{dt} = A\vec{x}, \tag{5.5}$$

where A is an $n \times n$ constant coefficient matrix. The general solution to the differential equation is:

$$\vec{x}(t) = e^{tA}\vec{C} = e^{tA} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}, \tag{5.6}$$

where $C_1, C_2, \cdots, C_n$ are arbitrary constants. The solution of this initial value problem is

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0 \tag{5.7}$$

$$\vec{x}(t) = e^{tA}\vec{x}_0. \tag{5.8}$$

From Equation (2), the expression for the time dependent vector can be re-expressed as

$$\vec{x}(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}. \tag{5.9}$$

Arbitrary constants C_n can be solved from the initial condition of the differential equation

$$\vec{x}_0 = \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \cdots & \vec{x}_n(0) \end{bmatrix} \vec{C} \tag{5.10}$$

$$\Rightarrow \vec{C} = \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \cdots & \vec{x}_n(0) \end{bmatrix}^{-1} \vec{x}_0. \tag{5.11}$$

Substituting Equation (7) back to Equation (5), the solution of the differential equation is given by

$$\vec{x}(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \cdots & \vec{x}_n(0) \end{bmatrix}^{-1} \vec{x}_0. \tag{5.12}$$

Comparing the expression derived above from Equation (4), e^{tA} is derived to be

$$\begin{aligned} e^{tA} &= \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \cdots & \vec{x}_n(0) \end{bmatrix}^{-1} \\ &= M(t)M(0)^{-1}. \end{aligned} \tag{5.13}$$

where $M(t)$ is the fundamental matrix solution.

5.2.2 Implementation of Sylvester's Formula

Consider a 4×4 matrix A with eigenvalues λ_1, λ_2, λ_3, such that λ_1 has a multiplicity of 2. Buchheim's generalisation to the Sylvester's Formula yields the following equality [29]

$$e^{tA} = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} + C_3 e^{\lambda_2 t} + C_4 e^{\lambda_3 t}, \tag{5.14}$$

where C_1, C_2, C_3, and C_4 are constants to be solved. To solve for the constants, the equality is differentiated thrice with respect to t, obtaining 3 other equations

$$Ae^{tA} = C_1\lambda_1 e^{\lambda_1 t} + C_2 e^{\lambda_1 t}(t\lambda_1 + 1) + C_3\lambda_2 e^{\lambda_2 t} + C_4\lambda_3 e^{\lambda_3 t} \tag{5.15}$$

$$A^2e^{tA} = C_1\lambda_1^2 e^{\lambda_1 t} + C_2\lambda_1 e^{\lambda_1 t}(t\lambda_1 + 2) + C_3\lambda_2^2 e^{\lambda_2 t} + C_4\lambda_3^2 e^{\lambda_3 t} \tag{5.16}$$

$$A^3e^{tA} = C_1\lambda_1^3 e^{\lambda_1 t} + C_2\lambda_1^2 e^{\lambda_1 t}(t\lambda_1 + 3) + C_3\lambda_2^3 e^{\lambda_2 t} + C_4\lambda_3^3 e^{\lambda_3 t}. \tag{5.17}$$

Setting $t = 0$ for the above four equations,

$$
\begin{aligned}
1 &= C_1 + C_2 + C_3 + C_4 & (5.18)\\
A &= C_1\lambda_1 + C_2\left(t\lambda_1 + 1\right) + C_3\lambda_2 + C_4\lambda_3 & (5.19)\\
A^2 &= C_1\lambda_1^2 + C_2\lambda_1\left(t\lambda_1 + 2\right) + C_3\lambda_2^2 + C_4\lambda_3^2 & (5.20)\\
A^3 &= C_1\lambda_1^3 + C_2\lambda_1^2\left(t\lambda_1 + 3\right) + C_3\lambda_2^3 + C_4\lambda_3^3. & (5.21)
\end{aligned}
$$

the constants can be solved and a matrix representation of the matrix exponential e^{tA} is derived.

5.3 Derivations of elements for Covariance Matrix

5.3.1 Ground State

For subsystem A,

$$
\begin{aligned}
V_{11} &= \frac{1}{2}\left\langle\{\Delta\bar{x}_A, \Delta\bar{x}_A\}\right\rangle\\
&= \frac{1}{2}\left\langle 2\bar{x}_A^2\right\rangle - \left\langle\bar{x}_A\right\rangle\left\langle\bar{x}_A\right\rangle\\
&= \langle 0|\frac{\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}}{2}|0\rangle - \langle 0|\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}|0\rangle\langle 0|\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}|0\rangle\\
&= \frac{1}{2} & (5.22)
\end{aligned}
$$

$$
\begin{aligned}
V_{22} &= \frac{1}{2}\left\langle\{\Delta\bar{p}_A, \Delta\bar{p}_A\}\right\rangle\\
&= \frac{1}{2}\left\langle 2\bar{p}_A^2\right\rangle - \left\langle\bar{p}_A\right\rangle\left\langle\bar{p}_A\right\rangle\\
&= \frac{1}{2}\langle 0|\frac{\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}}{-2i}|0\rangle - \langle 0|\frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}|0\rangle\langle 0|\frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}|0\rangle\\
&= \frac{1}{2} & (5.23)
\end{aligned}
$$

$$
\begin{aligned}
V_{12} &= \frac{1}{2}\left\langle\{\Delta\bar{x_A}, \Delta\bar{p_A}\}\right\rangle\\
&= \frac{1}{2}\left\langle\bar{x}_A\bar{p}_A + \bar{p}_A\bar{x}_A\right\rangle - \left\langle\bar{x}_A\right\rangle\left\langle\bar{p}_A\right\rangle\\
&= \frac{1}{2}\left\langle\frac{2\hat{a}^2 - 2\hat{a}^{\dagger 2}}{2i}\right\rangle\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
V_{21} &= \frac{1}{2}\left\langle\{\Delta\bar{p}_A, \Delta\bar{x}_A\}\right\rangle\\
&= \frac{1}{2}\left\langle\bar{p}_A\bar{x}_A + \bar{x}_A\bar{p}_A\right\rangle - \left\langle\bar{p}_A\right\rangle\left\langle\bar{x}_A\right\rangle
\end{aligned}
$$

$$= \quad V_{12} = 0. \tag{5.24}$$

For subsystem B,

$$\begin{aligned}
V_{33} &= \frac{1}{2}\langle\{\Delta\bar{x}_B, \Delta\bar{x}_B\}\rangle \\
&= \frac{1}{2}\langle 2\bar{x}_B^2\rangle - \langle\bar{x}_B\rangle\langle\bar{x}_B\rangle \\
&= \langle 0|\frac{\hat{b}^2 + \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} + \hat{b}^{\dagger 2}}{2}|0\rangle - \langle 0|\frac{\hat{b}+\hat{b}^\dagger}{\sqrt{2}}|0\rangle\langle 0|\frac{\hat{b}+\hat{b}^\dagger}{\sqrt{2}}|0\rangle \\
&= \frac{1}{2}
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
V_{44} &= \frac{1}{2}\langle\{\Delta\bar{p}_B, \Delta\bar{p}_B\}\rangle \\
&= \frac{1}{2}\langle 2\bar{p}_B^2\rangle - \langle\bar{p}_B\rangle\langle\bar{p}_B\rangle \\
&= \frac{1}{2}\langle 0|\frac{\hat{b}^2 - \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} + \hat{ab}^{\dagger 2}}{-2i}|0\rangle - \langle 0|\frac{\hat{b}-\hat{b}^\dagger}{i\sqrt{2}}|0\rangle\langle 0|\frac{\hat{b}-\hat{b}^\dagger}{i\sqrt{2}}|0\rangle \\
&= \frac{1}{2}
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
V_{34} &= \frac{1}{2}\langle\{\Delta\bar{x}_B, \Delta\bar{p}_B\}\rangle \\
&= \frac{1}{2}\langle\bar{x}_B\bar{p}_B + \bar{p}_B\bar{x}_B\rangle - \langle\bar{x}_B\rangle\langle\bar{p}_B\rangle \\
&= \frac{1}{2}\langle\frac{2\hat{b}^2 - 2\hat{b}^{\dagger 2}}{2i}\rangle \\
&= 0
\end{aligned}$$

$$\begin{aligned}
V_{21} &= \frac{1}{2}\langle\{\Delta\bar{p}_B, \Delta\bar{x}_B\}\rangle \\
&= \frac{1}{2}\langle\bar{p}_B\bar{x}_B + \bar{x}_B\bar{p}_B\rangle - \langle\bar{p}_B\rangle\langle\bar{x}_B\rangle \\
&= V_{12} = 0.
\end{aligned} \tag{5.27}$$

For intermodal correlations,

$\{x_{m,n}, p_{m,n}\}$ terms, where $m \neq n$:

$$\begin{aligned}
V_{mn} &= \frac{1}{2}\langle\bar{x}_m\bar{p}_n - \bar{p}_n\bar{x}_m\rangle - \langle\bar{x}_m\rangle\langle\bar{p}_n\rangle \\
&= \frac{1}{2}\left\langle\frac{1}{4i}(\hat{a}_m + \hat{a}_m^\dagger)(\hat{a}_n - \hat{a}_n^\dagger) - \frac{1}{4i}(\hat{a}_n - \hat{a}_n^\dagger)(\hat{a}_m + \hat{a}_m^\dagger)\right\rangle \\
&= \frac{1}{2}\left\langle\frac{1}{4i}(\hat{a}_m\hat{a}_n - \hat{a}_m\hat{a}_n^\dagger + \hat{a}_m^\dagger\hat{a}_n - \hat{a}_m^\dagger\hat{a}_n^\dagger) - \frac{1}{4i}(\hat{a}_n\hat{a}_m - \hat{a}_n\hat{a}_m^\dagger + \hat{a}_n^\dagger\hat{a}_m - \hat{a}_n^\dagger\hat{a}_m^\dagger)\right\rangle \\
&= 0.
\end{aligned} \tag{5.28}$$

$\{\bar{x}_m, \bar{x}_n\}$ terms, where $m \neq n$,

$$\begin{aligned} V_{mn} &= \frac{1}{2}\langle \bar{x}_m\bar{x}_n - \bar{x}_n\bar{x}_m\rangle - \langle \bar{x}_m\rangle\langle \bar{x}_n\rangle \\ &= \frac{1}{2}\left\langle \frac{1}{4}(\hat{a}_m + \hat{a}_m^\dagger)(\hat{a}_n + \hat{a}_n^\dagger) - \frac{1}{4}(\hat{a}_n + \hat{a}_n^\dagger)(\hat{a}_m + \hat{a}_m^\dagger)\right\rangle \\ &= \frac{1}{2}\left\langle \frac{1}{4}(\hat{a}_m\hat{a}_n + \hat{a}_m\hat{a}_n^\dagger + \hat{a}_m^\dagger\hat{a}_n + \hat{a}_m^\dagger\hat{a}_n^\dagger) - \frac{1}{4}(\hat{a}_n\hat{a}_m + \hat{a}_n\hat{a}_m^\dagger + \hat{a}_n^\dagger\hat{a}_m + \hat{a}_n^\dagger\hat{a}_m^\dagger)\right\rangle \\ &= 0. \end{aligned} \tag{5.29}$$

$\{\bar{p}_m, \bar{p}_n\}$ terms, where $m \neq n$,

$$\begin{aligned} V_{mn} &= \frac{1}{2}\langle \bar{p}_m\bar{p}_n - \bar{p}_n\bar{p}_m\rangle - \langle \bar{p}_m\rangle\langle \bar{p}_n\rangle \\ &= \frac{1}{2}\left\langle -\frac{1}{4}(\hat{a}_m - \hat{a}_m^\dagger)(\hat{a}_n - \hat{a}_n^\dagger) + \frac{1}{4}(\hat{a}_n - \hat{a}_n^\dagger)(\hat{a}_m + \hat{a}_m^\dagger)\right\rangle \\ &= \frac{1}{2}\left\langle -\frac{1}{4}(\hat{a}_m\hat{a}_n - \hat{a}_m\hat{a}_n^\dagger - \hat{a}_m^\dagger\hat{a}_n + \hat{a}_m^\dagger\hat{a}_n^\dagger) + \frac{1}{4}(\hat{a}_n\hat{a}_m - \hat{a}_n\hat{a}_m^\dagger - \hat{a}_n^\dagger\hat{a}_m + \hat{a}_n^\dagger\hat{a}_m^\dagger)\right\rangle \\ &= 0. \end{aligned} \tag{5.30}$$

5.3.2 Squeezed Ground State

Before the derivation for the covariance matrix of squeezed ground state, it is convenient to first derive the respective expectation values of operators.

Interactions between Squeezing Operators, Creation and Annihilation Operators

Evaluating the interactions between the squeezing operators, creation and annihilation operators,

$$\begin{aligned} \hat{S}^\dagger\hat{a}\hat{S} &= \exp\left(-\frac{\delta^*}{2}\hat{a}^2 + \frac{\delta}{2}\hat{a}^{\dagger^2}\right)\hat{a}\exp\left(-\frac{\delta}{2}\hat{a}^{\dagger^2} + \frac{\delta^*}{2}\hat{a}\right) \\ &= e^\mu\hat{a}e^{-\mu}. \end{aligned} \tag{5.31}$$

Invoking the Baker-Campbell-Hausdorff formula,

$$e^\mu\hat{a}e^{-\mu} = \hat{a} + [\mu,\hat{a}] + \frac{1}{2!}[\mu,[\mu,\hat{a}]] + \frac{1}{3!}[\mu,[\mu,[\mu,\hat{a}]]] + \cdots \tag{5.32}$$

$$\begin{aligned} [\mu,\hat{a}] &= \frac{1}{2}[\delta\hat{a}^{\dagger^2} - \delta^*\hat{a}^2, \hat{a}] \\ &= \frac{1}{2}\left(\delta\hat{a}^{\dagger^2}\hat{a} - \delta^*\hat{a}^3 - \delta\hat{a}\hat{a}^{\dagger^2} + \delta^*\hat{a}^3\right) \\ &= \frac{1}{2}\left(\delta[\hat{a}^{\dagger^2}, \hat{a}]\right) \\ &= \frac{\delta}{2}\left(\hat{a}^\dagger[\hat{a}^\dagger,\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a}^\dagger\right) \\ &= -\delta\hat{a}^\dagger \end{aligned} \tag{5.33}$$

$$
\begin{aligned}
[\mu,[\mu,\hat{a}]] &= \left[\frac{\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2}{2},-\delta\hat{a}^\dagger\right] \\
&= \frac{1}{2}[\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2,-\delta\hat{a}^\dagger] \\
&= \frac{1}{2}[-\delta^2\hat{a}^{\dagger^3}+|\delta|^2\hat{a}^2\hat{a}^\dagger+\delta^2\hat{a}^{\dagger^3}-|\delta|^2\hat{a}^2\hat{a}^\dagger] \\
&= \frac{1}{2}[|\delta|^2(\hat{a}^2\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^2)] \\
&= \frac{|\delta|^2}{2}[\hat{a}^2,\hat{a}^\dagger] \\
&= \frac{|\delta|^2}{2}(\hat{a}[\hat{a},\hat{a}^\dagger]+[\hat{a},\hat{a}^\dagger]\hat{a}) \\
&= \hat{a}|\delta|^2 \qquad (5.34)
\end{aligned}
$$

$$
\begin{aligned}
[\mu,[\mu,[\mu,\hat{a}]]] &= \left[\frac{\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2}{2},\hat{a}|\delta|^2\right] \\
&= \frac{1}{2}[\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2,\hat{a}|\delta|^2] \\
&= \frac{|\delta|^2}{2}[\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2,\hat{a}] \\
&= \frac{|\delta|^2}{2}(-2\delta\hat{a}^\dagger) \\
&= -\delta|\delta|^2\hat{a}^\dagger \qquad (5.35)
\end{aligned}
$$

$$
\begin{aligned}
[\mu,[\mu,[\mu,[\mu,\hat{a}]]]] &= \frac{1}{2}[\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2,-\delta|\delta|^2\hat{a}^\dagger] \\
&= \frac{\delta|\delta|^2}{2}[\delta\hat{a}^{\dagger^2}-\delta^*\hat{a}^2,-\delta\hat{a}^\dagger] \\
&= \frac{|\delta|^2}{2}(2\hat{a}|\delta|^2) \\
&= \hat{a}|\delta|^2|\delta|^2 \qquad (5.36) \\
&\vdots
\end{aligned}
$$

Combining the above few equations, we arrive at the following expression

$$
\begin{aligned}
\hat{S}^\dagger\hat{a}\hat{S} &= \hat{a}-\delta\hat{a}^\dagger+\frac{1}{2}\hat{a}|\delta|^2-\frac{1}{3!}\delta|\delta|^2\hat{a}^\dagger+\frac{1}{4!}\hat{a}|\delta|^2|\delta|^2+\cdots \\
&= \left(\hat{a}+\frac{1}{2}\hat{a}|\delta|^2+\frac{1}{4!}\hat{a}|\delta|^4+\cdots\right)-\left(\delta\hat{a}^\dagger+\frac{1}{3!}\delta|\delta|^2\hat{a}^\dagger+\cdots\right) \\
&= \hat{a}\left(1+\frac{1}{2}|\delta|^2+\frac{1}{4!}|\delta|^4+\cdots\right)-\hat{a}^\dagger\left(\delta+\frac{1}{3!}\delta|\delta|^2+\cdots\right) \\
&= \hat{a}\left(1+\frac{1}{2}r^2+\frac{1}{4!}r^4+\cdots\right)-\hat{a}^\dagger e^{i\phi}\left(r+\frac{1}{3!}r^3+\cdots\right) \\
&= \hat{a}\cosh(r)-\hat{a}^\dagger e^{i\phi}\sinh(r). \qquad (5.37)
\end{aligned}
$$

To get $\hat{S}\hat{a}\hat{S}^\dagger$, we simply replace δ with $-\delta$ in the above calculation:

$$\begin{aligned}
\hat{S}\hat{a}\hat{S}^\dagger &= \exp\left(-\frac{(-\delta)^*}{2}\hat{a}^2 + \frac{(-\delta)}{2}\hat{a}^{\dagger^2}\right)\hat{a}\exp\left(-\frac{(-\delta)}{2}\hat{a}^{\dagger^2} + \frac{(-\delta)^*}{2}\hat{a}\right) \\
&= \exp\left(-\frac{\delta}{2}\hat{a}^{\dagger^2} + \frac{\delta^*}{2}\hat{a}\right)\hat{a}\exp\left(-\frac{\delta^*}{2}\hat{a}^2 + \frac{\delta}{2}\hat{a}^{\dagger^2}\right) \\
&= \hat{a}\cosh(r) + \hat{a}^\dagger e^{i\phi}\sinh(r). \qquad (5.38)
\end{aligned}$$

Replacing $\hat{a}$ with $\hat{a}^\dagger$ in the derivation of Equation (30) and Equation (31), we obtain the following relations,

$$\hat{S}^\dagger\hat{a}^\dagger\hat{S} = \hat{a}^\dagger\cosh(r) - \hat{a}e^{-i\phi}\sinh(r) \qquad (5.39)$$

$$\hat{S}\hat{a}^\dagger\hat{S}^\dagger = \hat{a}^\dagger\cosh(r) + \hat{a}e^{-i\phi}\sinh(r). \qquad (5.40)$$

Expectation Values

Using the relationship from the previous section, the expectation values for squeezed ground state is calculated:

$$\begin{aligned}
\langle\bar{x}\rangle &= \mathrm{Tr}\left[\hat{S}^\dagger\bar{x}\hat{S}\rho\right] \\
&= \left\langle\hat{S}^\dagger\bar{x}\hat{S}\right\rangle \\
&= \frac{1}{\sqrt{2}}\left\langle\hat{S}^\dagger\hat{a}^\dagger\hat{S} + \hat{S}^\dagger\hat{a}\hat{S}\right\rangle \\
&= \frac{1}{\sqrt{2}}\left\langle\hat{a}^\dagger\cosh(r) - e^{-i\phi}\hat{a}\sinh(r) + \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\phi}\sinh(r)\right\rangle \\
&= \frac{1}{\sqrt{2}}\left\langle\cosh(r)(\hat{a}^\dagger + \hat{a})\right\rangle - \frac{1}{\sqrt{2}}\left\langle\sinh(r)(\hat{a}^\dagger e^{i\phi} + \hat{a}e^{-i\phi})\right\rangle \\
&= 0 \qquad (5.41)
\end{aligned}$$

$$\begin{aligned}
\langle\bar{p}\rangle &= \mathrm{Tr}\left[\hat{S}^\dagger\bar{p}\hat{S}\rho\right] \\
&= \left\langle\hat{S}^\dagger\bar{p}\hat{S}\right\rangle \\
&= \frac{1}{i\sqrt{2}}\left\langle\hat{S}^\dagger(\hat{a} - \hat{a}^\dagger)\hat{S}\right\rangle \\
&= \frac{1}{i\sqrt{2}}\left\langle\hat{S}^\dagger\hat{a}\hat{S} - \hat{S}^\dagger\hat{a}^\dagger\hat{S}\right\rangle \\
&= \frac{1}{i\sqrt{2}}\left\langle\hat{a}\cosh(r) - \hat{a}^\dagger e^{i\phi}\sinh(r) - \hat{a}^\dagger\cosh(r) + \hat{a}e^{-i\phi}\sinh(r)\right\rangle \\
&= \frac{1}{i\sqrt{2}}\left\langle\cosh(r)(\hat{a} - \hat{a}^\dagger\right\rangle + \frac{1}{i\sqrt{2}}\left\langle\sinh(r)(\hat{a}e^{-i\phi} - \hat{a}^\dagger e^{i\phi})\right\rangle \\
&= 0 \qquad (5.42)
\end{aligned}$$

$$\langle\bar{x}^2\rangle = \mathrm{Tr}\left[\hat{S}^\dagger\bar{x}^2\hat{S}\rho\right]$$

$$
\begin{aligned}
&= \left\langle \hat{S}^\dagger \bar{x}^2 \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger (\hat{a}^\dagger + \hat{a})^2 \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger (\hat{a}^{\dagger^2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}^2) \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a}^\dagger \hat{S} + \hat{S}^\dagger \hat{a} \hat{S} \hat{S}^\dagger \hat{a}^\dagger \hat{S} + \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a} \hat{S} + \hat{S}^\dagger \hat{a} \hat{S} \hat{S}^\dagger \hat{a} \hat{S} \right\rangle \\
&= \frac{1}{2} \langle \hat{a}^{\dagger^2} \cosh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{-i\phi} \cosh(r)\sinh(r) + \hat{a}^2 e^{-2i\phi} \sinh^2(r) \\
&\quad + \hat{a}\hat{a}^\dagger \cosh^2(r) - (\hat{a}^2 e^{-i\phi} + \hat{a}^{\dagger^2} e^{i\phi}) \cosh(r)\sinh(r) + \hat{a}^\dagger \hat{a} \sinh^2(r) \\
&\quad + \hat{a}^\dagger \hat{a} \cosh^2(r) - (\hat{a}^2 e^{-i\phi} + \hat{a}^{\dagger^2} e^{i\phi}) \cosh(r)\sinh(r) + \hat{a}\hat{a}^\dagger \sinh^2(r) \\
&\quad + \hat{a}^2 \cosh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{i\phi} \cosh(r)\sinh(r) + \hat{a}^{\dagger^2} e^{2i\phi} \sinh^2(r) \rangle \\
&= \frac{1}{2} \langle -(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{-i\phi} \cosh(r)\sinh(r) + \hat{a}\hat{a}^\dagger \cosh^2(r) + \hat{a}^\dagger \hat{a} \sinh^2(r) \\
&\quad + \hat{a}^\dagger \hat{a} \cosh^2(r) + \hat{a}\hat{a}^\dagger \sinh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{i\phi} \cosh(r)\sinh(r) \rangle \\
&= \frac{1}{2} \langle \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger \rangle \left\langle \cosh^2(r) + \sinh^2(r) - \cosh(r)\sinh(r) \left(e^{i\phi} + e^{-i\phi} \right) \right\rangle \\
&= \frac{1}{2} \left(\cosh(2r) - 2\cosh(r)\sinh(r)\cos(\phi) \right) \\
&= \frac{1}{2} (\cosh(2r) - \sinh(2r)\cos(\phi)) \qquad (5.43)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{p}^2 \rangle &= \mathrm{Tr} \left[\hat{S}^\dagger \bar{p}^2 \hat{S} \rho \right] \\
&= \left\langle \hat{S}^\dagger \bar{p}^2 \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger (\hat{a} - \hat{a}^\dagger)^2 \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger (\hat{a}^{\dagger^2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} + \hat{a}^2) \hat{S} \right\rangle \\
&= \frac{1}{2} \left\langle \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a}^\dagger \hat{S} - \hat{S}^\dagger \hat{a} \hat{S} \hat{S}^\dagger \hat{a}^\dagger \hat{S} - \hat{S}^\dagger \hat{a}^\dagger \hat{S} \hat{S}^\dagger \hat{a} \hat{S} + \hat{S}^\dagger \hat{a} \hat{S} \hat{S}^\dagger \hat{a} \hat{S} \right\rangle \\
&= \frac{1}{2} \langle \hat{a}^{\dagger^2} \cosh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{-i\phi} \cosh(r)\sinh(r) + \hat{a}^2 e^{-2i\phi} \sinh^2(r) \\
&\quad - \hat{a}\hat{a}^\dagger \cosh^2(r) + (\hat{a}^2 e^{-i\phi} + \hat{a}^{\dagger^2} e^{i\phi}) \cosh(r)\sinh(r) - \hat{a}^\dagger \hat{a} \sinh^2(r) \\
&\quad - \hat{a}^\dagger \hat{a} \cosh^2(r) + (\hat{a}^2 e^{-i\phi} + \hat{a}^{\dagger^2} e^{i\phi}) \cosh(r)\sinh(r) - \hat{a}\hat{a}^\dagger \sinh^2(r) \\
&\quad + \hat{a}^2 \cosh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{i\phi} \cosh(r)\sinh(r) + \hat{a}^{\dagger^2} e^{2i\phi} \sinh^2(r) \rangle \\
&= \frac{1}{2} \langle -(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{-i\phi} \cosh(r)\sinh(r) - \hat{a}\hat{a}^\dagger \cosh^2(r) - \hat{a}^\dagger \hat{a} \sinh^2(r) \\
&\quad - \hat{a}^\dagger \hat{a} \cosh^2(r) - \hat{a}\hat{a}^\dagger \sinh^2(r) - (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) e^{i\phi} \cosh(r)\sinh(r) \rangle \\
&= \frac{1}{2} \langle \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger \rangle \left\langle \cosh^2(r) + \sinh^2(r) + \cosh(r)\sinh(r) \left(e^{i\phi} + e^{-i\phi} \right) \right\rangle \\
&= \frac{1}{2} \left(\cosh(2r) + 2\cosh(r)\sinh(r)\cos(\phi) \right) \\
&= \frac{1}{2} (\cosh(2r) + \sinh(2r)\cos(\phi)) \qquad (5.44)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{x}\bar{p} + \bar{p}\bar{x} \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger(\bar{x}\bar{p}+\bar{p}\bar{x})\hat{S}\rho_{th}\right] \\
&= \frac{1}{2i}\left\langle \hat{S}^\dagger[(\hat{a}^\dagger+\hat{a})(\hat{a}-\hat{a}^\dagger)+(\hat{a}-\hat{a}^\dagger)(\hat{a}^\dagger+\hat{a})]\hat{S}\right\rangle \\
&= \frac{1}{2i}\left\langle \hat{S}^\dagger\left(2\hat{a}^2 - 2\hat{a}^{\dagger^2}\right)\hat{S}\right\rangle \\
&= \frac{1}{i}\left\langle \hat{S}^\dagger\left(\hat{a}^2 - \hat{a}^{\dagger^2}\right)\hat{S}\right\rangle \\
&= \frac{1}{i}\left\langle -e^{i\phi}\cosh(r)\sinh(r)(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger) + e^{-i\phi}\cosh(r)\sinh(r)(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)\right\rangle \\
&= \frac{1}{i}\langle \hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\rangle\left(\cosh(r)\sinh(r)(-2i\sinh(\phi)\right) \\
&= -\cosh(r)\sinh(r)\sin(\phi). \qquad (5.45)
\end{aligned}
$$

Covariance Matrix

By using $(r_A, \phi_A), (r_B, \phi_B)$ for different squeezing operations on the respective ground states harmonic oscillators, we can derive the terms for the covariance matrix at $\omega t = 0$. Similar to the case for ground state, the non-zero terms of the covariance matrix are local mode correlation for the respective subsystem.
For subsystem A,

$$
\begin{aligned}
V_{11} &= \frac{1}{2}\langle\{\Delta\bar{x}_A, \Delta\bar{x}_A\}\rangle \\
&= \frac{1}{2}\langle 2\bar{x}_A^2\rangle - \langle\bar{x}_A\rangle\langle\bar{x}_A\rangle \\
&= \frac{1}{2}(\cosh(2r_A) - \sinh(2r_A)\cos(\phi_A)) \qquad (5.46)
\end{aligned}
$$

$$
\begin{aligned}
V_{22} &= \frac{1}{2}\langle\{\Delta\bar{p}_A, \Delta\bar{p}_A\}\rangle \\
&= \frac{1}{2}\langle 2\bar{p}_A^2\rangle - \langle\bar{p}_A\rangle\langle\bar{p}_A\rangle \\
&= \frac{1}{2}(\cosh(2r_A) + \sinh(2r_A)\cos(\phi_A)) \qquad (5.47)
\end{aligned}
$$

$$
\begin{aligned}
V_{12} &= \frac{1}{2}\langle\{\Delta\bar{x}_A, \Delta\bar{p}_A\}\rangle \\
&= \frac{1}{2}\langle \bar{x}_A\bar{p}_A + \bar{p}_A\bar{x}_A\rangle - \langle\bar{x}_A\rangle\langle\bar{p}_A\rangle \\
&= -\cosh(r_A)\sinh(r_A)\sin(\phi_A) \qquad (5.48)
\end{aligned}
$$

$$
\begin{aligned}
V_{21} &= \frac{1}{2}\langle\{\Delta\bar{p}_A, \Delta\bar{x}_A\}\rangle \\
&= \frac{1}{2}\langle \bar{p}_A\bar{x}_A + \bar{x}_A p_A\rangle - \langle\bar{p}_A\rangle\langle\bar{x}_A\rangle \\
&= -\cosh(r_A)\sinh(r_A)\sin(\phi_A). \qquad (5.49)
\end{aligned}
$$

For subsystem B,

$$\begin{aligned}
V_{33} &= \frac{1}{2}\left\langle\{\Delta\bar{x}_B, \Delta\bar{x}_B\}\right\rangle \\
&= \frac{1}{2}\left\langle 2\bar{x}_B^2\right\rangle - \left\langle\bar{x}_B\right\rangle\left\langle\bar{x}_B\right\rangle \\
&= \frac{1}{2}(\cosh(2r_B) - \sinh(2r_B)\cos(\phi_B))
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
V_{44} &= \frac{1}{2}\left\langle\{\Delta\bar{p}_B, \Delta\bar{p}_B\}\right\rangle \\
&= \frac{1}{2}\left\langle 2\bar{p}_B^2\right\rangle - \left\langle\bar{p}_B\right\rangle\left\langle\bar{p}_B\right\rangle \\
&= \frac{1}{2}(\cosh(2r_B) + \sinh(2r_B)\cos(\phi_B))
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
V_{34} &= \frac{1}{2}\left\langle\{\Delta\bar{x}_B, \Delta\bar{p}_B\}\right\rangle \\
&= \frac{1}{2}\left\langle\bar{x}_B\bar{p}_B + \bar{p}_B\bar{x}_B\right\rangle - \left\langle\bar{x}_B\right\rangle\left\langle\bar{p}_B\right\rangle \\
&= -\cosh(r_B)\sinh(r_B)\sin(\phi_B)
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
V_{43} &= \frac{1}{2}\left\langle\{\Delta\bar{p}_B, \Delta\bar{x}_B\}\right\rangle \\
&= \frac{1}{2}\left\langle\bar{p}_B\bar{x}_B + \bar{x}_B\bar{p}_B\right\rangle - \left\langle\bar{p}_B\right\rangle\left\langle\bar{x}_B\right\rangle \\
&= -\cosh(r_B)\sinh(r_B)\sin(\phi_B).
\end{aligned} \tag{5.53}$$

5.3.3 Thermal State

Expectation Values

Initialising the harmonic oscillators in thermal states, the expectation values for the operators are

$$\begin{aligned}
\langle\bar{x}\rangle &= \mathrm{Tr}[\bar{x}\rho_{th}] \\
&= \mathrm{Tr}\left[\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\frac{1}{N+1}\sum_{m=0}\left(\frac{N}{N+1}\right)^m |m\rangle\langle m|\right] \\
&= \mathrm{Tr}\left[\frac{1}{\sqrt{2}(N+1)}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c|m-1\rangle + a_{c^*}|m+1\rangle)\langle m|\right] \\
&= \frac{1}{\sqrt{2}(N+1)}\sum_{k=0}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c\delta_{k,m-1} + a_{c^*}\delta_{k,m+1})\delta_{m,k} \\
&= 0
\end{aligned} \tag{5.54}$$

$$\langle\bar{p}\rangle = \mathrm{Tr}[\bar{p}\rho_{th}]$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[\frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}\frac{1}{N+1}\sum_{m=0}\left(\frac{N}{N+1}\right)^m |m\rangle\langle m|\right] \\
&= \mathrm{Tr}\left[\frac{1}{i\sqrt{2}(N+1)}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c|m-1\rangle - a_{c^*}|m+1\rangle)\langle m|\right] \\
&= \frac{1}{i\sqrt{2}(N+1)}\sum_{k=0}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c\delta_{k,m-1} - a_{c^*}\delta_{k,m+1})\delta_{m,k} \\
&= 0
\end{aligned} \tag{5.55}
$$

$$
\begin{aligned}
\langle \bar{x}^2\rangle &= \left\langle \frac{\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}}{2}\right\rangle \\
&= \left\langle \frac{\hat{a}^2+\hat{a}^{\dagger^2}}{2}\right\rangle + \left\langle \frac{\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}}{2}\right\rangle \\
&= \left\langle \frac{2\hat{a}^\dagger\hat{a}+1}{2}\right\rangle \quad , \ \because [\hat{a},\hat{a}^\dagger]=1 \\
&= \frac{1}{2}+\langle\hat{a}^\dagger\hat{a}\rangle \\
&= \frac{1}{2}+N \quad , \ \because \langle\hat{a}^\dagger\hat{a}\rangle = \langle n\rangle
\end{aligned} \tag{5.56}
$$

$$
\begin{aligned}
\langle \bar{p}^2\rangle &= \left\langle \frac{\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}}{-2}\right\rangle \\
&= \left\langle \frac{-\hat{a}^2-\hat{a}^{\dagger^2}}{2}\right\rangle + \left\langle \frac{\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}}{2}\right\rangle \\
&= \frac{1}{2}+N
\end{aligned} \tag{5.57}
$$

$$
\begin{aligned}
\langle \bar{p}\bar{x}+\bar{x}\bar{p}\rangle &= \left\langle \frac{\hat{a}^2-\hat{a}^{\dagger^2}}{i}\right\rangle \\
&= \mathrm{Tr}\left[\frac{\hat{a}^2-\hat{a}^{\dagger^2}}{i}\frac{1}{N+1}\sum_{m=0}\left(\frac{N}{N+1}\right)^m |m\rangle\langle m|\right] \\
&= \mathrm{Tr}\left[\frac{1}{i(N+1)}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c a_{c-1}|m-2\rangle - a_{c*}a_{(c+1)^*}|m+2\rangle)\langle m|\right] \\
&= \frac{1}{i(N+1)}\sum_{k=0}\sum_{m=0}\left(\frac{N}{N+1}\right)^m (a_c a_{c-1}\delta_{k,m-2} - a_{c*}a_{(c+1)^*}\delta_{k,m+2})\delta_{k,m} \\
&= 0,
\end{aligned} \tag{5.58}
$$

where a_c, a_{c-1} and a_{c^*}, $a_{(c+1)^*}$ are the coefficients of creation and annihilation operators respectively.

Covariance Matrix

Subjecting the two oscillators to thermal environment of T_A and T_B respectively, the covariance matrix of the initial system is constructed as follows:

For subsystem A,

$$\begin{aligned} V_{11} &= \frac{1}{2}\left\langle 2\bar{x}_A^2 \right\rangle - \left\langle \bar{x}_A \right\rangle \left\langle \bar{x}_A \right\rangle \\ &= \frac{1}{2} + N_A \end{aligned} \tag{5.59}$$

$$\begin{aligned} V_{22} &= \frac{1}{2}\left\langle 2\bar{p}_A^2 \right\rangle - \left\langle \bar{p}_A \right\rangle \left\langle \bar{p}_A \right\rangle \\ &= \frac{1}{2} + N_A \end{aligned} \tag{5.60}$$

$$\begin{aligned} V_{12} &= \frac{1}{2}\left\langle \hat{x}_A\hat{p}_A + \hat{p}_A\hat{x}_A \right\rangle - \left\langle \hat{x}_A \right\rangle \left\langle \hat{p}_A \right\rangle \\ &= 0 \end{aligned} \tag{5.61}$$

$$\begin{aligned} V_{21} &= \frac{1}{2}\left\langle \hat{p}_A\hat{x}_A + \hat{x}_A\hat{p}_A \right\rangle - \left\langle \hat{p}_A \right\rangle \left\langle \hat{x}_A \right\rangle \\ &= 0. \end{aligned} \tag{5.62}$$

For subsystem B,

$$\begin{aligned} V_{33} &= \frac{1}{2}\left\langle 2\bar{x}_B^2 \right\rangle - \left\langle \bar{x}_B \right\rangle \left\langle \bar{x}_B \right\rangle \\ &= \frac{1}{2} + N_B \end{aligned} \tag{5.63}$$

$$\begin{aligned} V_{44} &= \frac{1}{2}\left\langle 2\bar{p}_B^2 \right\rangle - \left\langle \bar{p}_B \right\rangle \left\langle \bar{p}_B \right\rangle \\ &= \frac{1}{2} + N_B \end{aligned} \tag{5.64}$$

$$\begin{aligned} V_{34} &= \frac{1}{2}\left\langle \hat{x}_B\hat{p}_B + \hat{p}_B\hat{x}_B \right\rangle - \left\langle \hat{x}_B \right\rangle \left\langle \hat{p}_B \right\rangle \\ &= 0 \end{aligned} \tag{5.65}$$

$$\begin{aligned} V_{43} &= \frac{1}{2}\left\langle \hat{p}_B\hat{x}_B + \hat{x}_B\hat{p}_B \right\rangle - \left\langle \hat{p}_B \right\rangle \left\langle \hat{x}_B \right\rangle \\ &= 0. \end{aligned} \tag{5.66}$$

The elements of the intermodal correlation are zero with the same proof as shown in previous section.

5.3.4 Squeezed Thermal State

Expectation Values

By squeezing thermal states of oscillators, we derive a set of expectation values for the operators:

$$
\begin{aligned}
\langle \bar{x} \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger \bar{x} \hat{S} \rho_{th}\right] \\
&= \left\langle \hat{S}^\dagger \bar{x} \hat{S} \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle \hat{S}^\dagger \hat{a}^\dagger \hat{S} + \hat{S}^\dagger \hat{a} \hat{S} \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle \hat{a}^\dagger \cosh(r) - e^{-i\phi} \hat{a} \sinh(r) + \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\phi} \sinh(r) \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle \cosh(r)(\hat{a}^\dagger + \hat{a}) \right\rangle - \frac{1}{\sqrt{2}} \left\langle \sinh(r)(\hat{a}^\dagger e^{i\phi} + \hat{a} e^{-i\phi}) \right\rangle \\
&= 0 \qquad (5.67)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{p} \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger \bar{p} \hat{S} \rho_{th}\right] \\
&= \left\langle \hat{S}^\dagger \bar{p} \hat{S} \right\rangle \\
&= \frac{1}{i\sqrt{2}} \left\langle \hat{S}^\dagger (\hat{a} - \hat{a}^\dagger) \hat{S} \right\rangle \\
&= \frac{1}{i\sqrt{2}} \left\langle \hat{S}^\dagger \hat{a} \hat{S} - \hat{S}^\dagger \hat{a}^\dagger \hat{S} \right\rangle \\
&= \frac{1}{i\sqrt{2}} \left\langle \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\phi} \sinh(r) - \hat{a}^\dagger \cosh(r) + \hat{a} e^{-i\phi} \sinh(r) \right\rangle \\
&= \frac{1}{i\sqrt{2}} \left\langle \cosh(r)(\hat{a} - \hat{a}^\dagger \right\rangle + \frac{1}{i\sqrt{2}} \left\langle \sinh(r)(\hat{a} e^{-i\phi} - \hat{a}^\dagger e^{i\phi}) \right\rangle \\
&= 0 \qquad (5.68)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{x}^2 \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger \bar{x}^2 \hat{S} \rho_{th}\right] \\
&= \frac{1}{2} \langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \rangle \left\langle \cosh^2(r) + \sinh^2(r) - \cosh(r) \sinh(r) \left(e^{i\phi} + e^{-i\phi}\right) \right\rangle \\
&= \frac{2N+1}{2} \left(\cosh(2r) - 2\cosh(r) \sinh(r) \cos(\phi)\right) \\
&= \frac{2N+1}{2} (\cosh(2r) - \sinh(2r) \cos(\phi) \qquad (5.69)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{p}^2 \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger \bar{p}^2 \hat{S} \rho_{th}\right] \\
&= \frac{1}{2} \langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \rangle \left\langle \cosh^2(r) + \sinh^2(r) + \cosh(r) \sinh(r) \left(e^{i\phi} + e^{-i\phi}\right) \right\rangle \\
&= \frac{2N+1}{2} \left(\cosh(2r) + 2\cosh(r) \sinh(r) \cos(\phi)\right) \\
&= \frac{2N+1}{2} (\cosh(2r) + \sinh(2r) \cos(\phi)) \qquad (5.70)
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{x}\bar{p} + \bar{p}\bar{x} \rangle &= \mathrm{Tr}\left[\hat{S}^\dagger (\bar{x}\bar{p} + \bar{p}\bar{x}) \hat{S} \rho_{th}\right] \\
&= \frac{1}{i} \langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \rangle \left(\cosh(r) \sinh(r) (-2i \sinh(\phi)\right) \\
&= -(2N+1) \cosh(r) \sinh(r) \sin(\phi). \qquad (5.71)
\end{aligned}
$$

Covariance Matrix

Squeezed thermal states with squeezing strength r and squeezing phase ϕ, the covariance matrix of the initial system is For subsystem A,

$$\begin{aligned} V_{11} &= \frac{1}{2}\langle\{\Delta\bar{x}_A, \Delta\bar{x}_A\}\rangle \\ &= \frac{1}{2}\langle 2x_A^2\rangle - \langle\bar{x}_A\rangle\langle\bar{x}_A\rangle \quad \text{, where } \Delta\bar{x} = \bar{x} - \langle\bar{x}\rangle \\ &= \frac{2N_A+1}{2}(\cosh(2r_A) - \sinh(2r_A)\cos(\phi_A)) \end{aligned} \tag{5.72}$$

$$\begin{aligned} V_{22} &= \frac{1}{2}\langle\{\Delta\bar{p}_A, \Delta\bar{p}_A\}\rangle \\ &= \frac{1}{2}\langle 2\bar{p}_A^2\rangle - \langle\bar{p}_A\rangle\langle\bar{p}_A\rangle \quad \text{, where } \Delta\bar{p} = \bar{p} - \langle\bar{p}\rangle \\ &= \frac{2N_A+1}{2}(\cosh(2r_A) + \sinh(2r_A)\cos(\phi_A)) \end{aligned} \tag{5.73}$$

$$\begin{aligned} V_{12} &= \frac{1}{2}\langle\{\Delta\bar{x}_A, \Delta\bar{p}_A\}\rangle \\ &= \frac{1}{2}\langle\bar{x}_A\bar{p}_A + \bar{p}_A\bar{x}_A\rangle - \langle\bar{x}_A\rangle\langle\bar{p}_A\rangle \\ &= -(2N_A+1)\cosh(r_A)\sinh(r_A)\sin(\phi_A) \end{aligned} \tag{5.74}$$

$$\begin{aligned} V_{21} &= \frac{1}{2}\langle\{\Delta\bar{p}_A, \Delta\bar{x}_A\}\rangle \\ &= \frac{1}{2}\langle\bar{p}_A\bar{x}_A + \bar{x}_A\bar{p}_A\rangle - \langle\bar{p}_A\rangle\langle\bar{x}_A\rangle \\ &= -(2N_A+1)\cosh(r_A)\sinh(r_A)\sin(\phi_A). \end{aligned} \tag{5.75}$$

For subsystem B,

$$\begin{aligned} V_{33} &= \frac{1}{2}\langle\{\Delta\bar{x}_B, \Delta\bar{x}_B\}\rangle \\ &= \frac{1}{2}\langle 2\bar{x}_B^2\rangle - \langle\bar{x}_B\rangle\langle\bar{x}_B\rangle \\ &= \frac{2N_B+1}{2}(\cosh(2r_B) - \sinh(2r_B)\cos(\phi_B)) \end{aligned} \tag{5.76}$$

$$\begin{aligned} V_{44} &= \frac{1}{2}\langle\{\Delta\bar{p}_B, \Delta\bar{p}_B\}\rangle \\ &= \frac{1}{2}\langle 2\bar{p}_B^2\rangle - \langle\bar{p}_B\rangle\langle\bar{p}_B\rangle \\ &= \frac{2N_B+1}{2}(\cosh(2r_B) + \sinh(2r_B)\cos(\phi_B)) \end{aligned} \tag{5.77}$$

$$V_{34} = \frac{1}{2}\langle\{\Delta\bar{x}_B, \Delta\bar{p}_B\}\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}\langle \bar{x}_B\bar{p}_B + \bar{p}_B\bar{x}_B\rangle - \langle \bar{x}_B\rangle\langle \bar{p}_B\rangle \\
&= -(2N_B+1)\cosh(r_B)\sinh(r_B)\sin(\phi_B) \qquad (5.78)
\end{aligned}
$$

$$
\begin{aligned}
V_{43} &= \frac{1}{2}\langle\{\Delta\bar{p}_B, \Delta\bar{x}_B\}\rangle \\
&= \frac{1}{2}\langle \bar{p}_B\bar{x}_B + \bar{x}_B\bar{p}_B\rangle - \langle \bar{p}_B\rangle\langle \bar{x}_B\rangle \\
&= -(2N_B+1)\cosh(r_B)\sinh(r_B)\sin(\phi_B). \qquad (5.79)
\end{aligned}
$$

5.4 Condition for Maximal Entanglement of Bipartite System

Entanglement can be maximised by the variation of $\sum(V)$.

$$
\begin{aligned}
\frac{d\tilde{v}^-}{d\sum(V)} &= \frac{d}{d\sum(V)}\left(\frac{\sum(V) - \left(\sum(V)^2 - \frac{1}{4}\right)^{1/2}}{2}\right)^{1/2} \\
&= \frac{\sqrt{2}}{4}\left(\frac{1}{\sqrt{\sum(V) - \sqrt{\sum(V)^2 - \frac{1}{4}}}}\right)\left(\frac{\sqrt{\sum(V)^2 - \frac{1}{4}} - \sum(V)}{\sqrt{\sum(V)^2 - \frac{1}{4}}}\right) \\
&= -\frac{\sqrt{2}}{4}\left(\frac{\sqrt{\sum(V) - \sqrt{\sum(V)^2 - \frac{1}{4}}}}{\sqrt{\sum(V)^2 - \frac{1}{4}}}\right). \qquad (5.80)
\end{aligned}
$$

Note that $\frac{d\tilde{v}^-}{d\sum(V)}$is stricty decreasing. Hence, in order to maximise entanglement, $\tilde{v}^-$ need to be minimise, i.e. $\sum(V)$ is to be maximised.

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