ENTANGLEMENT DISTRIBUTION BETWEEN TWO SEPARATE SYSTEMS VIA A THIRD SYSTEM

SUBMITTED
BY

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"The greatest challenge to any thinker is stating the problem in a way that will allow a solution."

Bertrand Russell

"We should be taught not to wait for inspiration to start a thing. Action always generates inspiration. Inspiration seldom generates action."

Frank Tibolt

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Abstract

Entanglement is a quantum correlation between pairs (or groups) of systems. It is a resource of crucial importance for quantum information processing such as quantum cryptography and quantum computing. In this thesis, we investigated entanglement distribution between two separate systems A and B using a third system C as an ancilla. For the case where C is allowed to interact discretely with A and then B, we found entanglement inequality: $E_{A-BC} - E_{B-AC} \leq E_{C-AB}$ in pure states. We also found violations of such inequality that will lead to excessive distribution in which entanglement distributed is higher than entanglement communicated. For the case where C is allowed to interact continuously with A and B, we proved that entanglement in the partition $A - B$ cannot increase under the condition that the state of the whole system is pure and the state of C is separable from AB. If the system evolves under the Born approximation, entanglement also cannot grow. This suggests that correlation in the partition $C - AB$ is needed to distribute entanglement. In pure states, we computed entanglement as a function of time with perturbation theory. We observed an artefact that entanglement gain is possible with separable C. We also computed entanglement in weak and strong interaction limit. In weak interaction, we found that maximum entanglement is achieved at the same time for weaker interaction strength at the expense of its value. However, in strong interaction the same value of maximum entanglement can be achieved for weaker interaction strength at the expense of time.

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Chapter 1

Introduction

1.1 Motivations of Research

Entanglement is a quantum correlation between systems in which the state of one is related to the others in a non-classical way. It was first recognised by Einstein *et al.* [1] and Schrödinger [2] and now has become a resource as real as energy [3]. This resource enables crucial quantum information processing such as quantum cryptography [4], quantum dense coding [5], and quantum teleportation [6]. Therefore it is important to have entanglement between systems in separate laboratories. A straightforward way to achieve it would be to entangle two systems in one laboratory and separate them after. Another way of achieving it is to involve an ancillary system (see FIG. 1.1): Entangle the ancilla to one system and send it to the other. Entanglement swapping [7] can then be applied to achieve entanglement between the two separate systems. These two ways of distributing entanglement will of course require an expensive quantum channel between laboratories so that entanglement does not undergo decoherence. Surprisingly, it has been shown by Cubitt *et al.* [8] that distributing entanglement using an ancillary system is possible without actually communicating entanglement. In other words,

C is separable from AB at all times. There are two ways: One is to interact the ancilla discretely (in a short amount of time) with A and then B. The other way is to make the ancilla interact with A and B continuously. We shall refer to the former and the latter as discrete and continuous interaction case respectively.

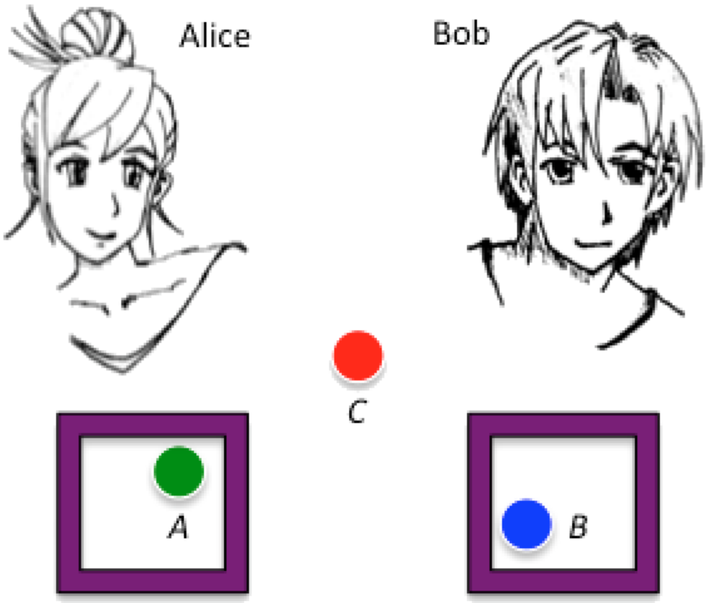

FIGURE 1.1: Setup of entanglement distribution between two separate systems, A and B, using an ancillary system, C.

In the discrete interaction case, if the state of the system is mixed entanglement gain can be more than entanglement communicated, in fact the entanglement communicated can be zero [8, 9]. If the state is pure there is a question whether such scenario is possible. Recently, a bound on the distributed entanglement which is given by quantum discord has also been derived [10, 11] and experimentally tested [12, 13]. However, a similar bound in the continuous interaction case has not been achieved. In particular one needs to establish that non-classical correlation in the partition $C - AB$ is needed to entangle A and B. Furthermore, it is also important to observe properties such as maximum entanglement and time to achieve it for varying values of interaction strength.

1.2 Research Objectives

In this thesis, first we investigated entanglement distribution in the discrete interaction case for pure states. The objective was to obtain a bound on the distributed entanglement and see whether excessive distribution, where the distributed entanglement is more than the communicated entanglement, was possible. We also investigated entanglement distribution in the continuous interaction case. The objective was to determine whether non-classical correlation in the partition $C-AB$ was really the key to distribute entanglement in the partition $A-B$. We also aimed to study how properties such as maximum entanglement and time required to accomplish it were changing with interaction strength.

1.3 Scope of Research

This thesis concerned the study of entanglement distribution between two separate systems using an ancillary system without the presence of background noise. In the discrete interaction case, entanglement measure investigated includes: entropy of entanglement, linear entropy, negativity, and logarithmic negativity. In the continuous case, relative entropy, entropy of entanglement, and negativity were used as entanglement measures. A specific example of Hamiltonian, used also by Cubitt *et al.* [8] as motivated by the ion-trap and cavity-QED experiments [14, 15], was used in the computation to evolve the state and hence obtain entanglement as a function of time. Bellow I will explain what is in this thesis and how it is organised.

Chapter 2 and 3 will serve mainly as reviews of non-classical correlations and entanglement distribution methods. Section 2.1 and 2.2 consider non-classical correlations and their measures that will be used in this thesis. Section 3.1 reviews entanglement distribution methods while the protocols will be introduced in Section 3.2. Chapter 2 and 3 also serve to establish notations used throughout the thesis. Some original material is included to fill gaps in literature.

Chapter 4 considers entanglement inequality and distribution for pure states in the discrete interaction case. Section 4.1 presents entanglement inequality for some entanglement measures in some dimensions. Violations of the inequality that will lead to excessive distribution will also be presented. This entanglement inequality is then used in Section 4.2 to obtain a bound on the distributed entanglement where the discrete interaction method is applied a number of times to maximise entanglement gain.

Chapter 5 and 6 consider entanglement distribution in the continuous interaction case. Section 5.1 presents the proof that entanglement in the partition $A - B$ cannot grow if the state of the system is pure and the state of C is separable from AB while Section 5.2 proves that entanglement also cannot grow if the system evolves under the Born approximation. Section 6.1 presents entanglement as a function time with perturbation theory and discusses an artefact created. Section 6.2 considers entanglement as a function of time without perturbation theory. It is divided into two subsections: 6.2.1 presents entanglement as a function of time for different values of interaction strength in weak interaction limit while 6.2.2 presents the one in strong interaction limit.

Appendix A contains derivation of Trotter expansion that is useful in Chapter 4.

Chapter 2

Non-classical Correlations and Measures

2.1 Non-classical Correlations

This Section contains brief introduction to non-classical correlations, i.e. quantum entanglement and quantum discord. First the definition and essential properties of quantum entanglement that are useful for this thesis will be mentioned. Next we will briefly review quantum discord and its relation to quantum entanglement.

2.1.1 Quantum Entanglement

Quantum systems differ from classical ones in that they can be correlated in a non-classical way. Entanglement is one of their distinctive features that has been studied and explored since it was first recognised [3]. Simple examples of pure entangled states are the Bell states:

$$\begin{aligned}
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right) \\
\left|\Phi^-\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B\right) \\
\left|\Psi^+\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right) \\
\left|\Psi^-\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)
\end{aligned} \tag{2.1}$$

These states allow non-classical correlations between systems A and B. If systems A and B are in different laboratories, entangled states cannot be prepared with local operations and classical communication (LOCC) [3]. However it can be prepared in one lab and later send one system to the other lab. Alternatively a third system can be used to distribute entanglement between the two separate systems; more on this entanglement distribution method will be reviewed in Chapter 3. If the state of the systems is not entangled, it can be written as follows

$$\sigma = \sum_i p_i \, \Pi_A^i \otimes \Pi_B^i \tag{2.2}$$

where p_is are probabilities and Πs are projectors. σ is also called separable state and it can be prepared with LOCC. In pure states, separability implies product states. In other words, the state of the system can be written as $|\Psi\rangle_{AB} = |\psi\rangle_A |\psi\rangle_B$

2.1.2 Quantum Discord

Apart from entanglement, discord is another form of non-classical correlation. It was introduced by Ollivier *et al.* [16] realising that two expressions of mutual information that are classically the same would differ if the system is quantum. If the state of AB has zero discord, then a measurement on one system, say $\{\Pi_A^i\}$, will reveal all information about system AB. The procedure will not perturb the state of AB. In contrast, if the system has non-zero discord then the state of AB is perturbed by all local operations. So it has been used as a quantification of how "quantum" a system is. Separability explained in the previous subsection does not usually result in vanishing discord. It is possible to have a state that has no entanglement and non-zero discord at the same time [17]. In pure states, discord measures entanglement [18].

2.2 Measures

This section considers some entanglement measures both for pure and mixed states that will be used in this thesis. Definitions and basic principles will be reviewed. Entropy of entanglement, linear entropy, negativity, and logarithmic negativity will be used in Chapter 4, while relative entropy, entropy of entanglement, and negativity will be used in Chapter 5 and 6. Examples of these measures on some states especially negativity in a system of three qubits will be presented. This negativity formula will be used in Chapter 4.

2.2.1 Entropy of Entanglement and Linear Entropy

If the state of AB is pure, Schmidt decomposition guarantees that the state can always be written as

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}\ |a\rangle^i |b\rangle^i \tag{2.3}$$

where $\{|a\rangle^i, |b\rangle^i\}$ are orthogonal bases of A and B respectively. The von Neumann entropy, defined as $-\mathrm{tr}\rho\log\rho$, of the whole system is zero because the state is pure. However if system A and B are entangled, the reduced state of A will appear as a mixed state and so will the reduced state of B. Entropy of entanglement measures how mixed the reduced systems are. *It is used to measure entanglement in pure* *states* [19, 20]. It is defined as follows

$$S_N \equiv -\mathrm{tr}(\rho_A \log \rho_A) = -\mathrm{tr}(\rho_B \log \rho_B) = -\sum_i p_i \log p_i \tag{2.4}$$

Bell states are maximally entangled; entropy of entanglement of all Bell states equal to unity. If the state of the system is separable, the entropy of entanglement is zero.

An approximation to entropy of entanglement is known as linear entropy. It is defined as

$$S_L \equiv 1 - \mathrm{tr}(\rho_A^2) = 1 - \mathrm{tr}(\rho_B^2) \tag{2.5}$$

It is easy to compute as it does not require diagonalisation of the density matrix. It has been used as an entanglement measure in two-fermion system [21].

2.2.2 Relative Entropy

Relative entropy is an entanglement measure for general states [17]. It is defined as follows

$$E(\rho) = \min_{\sigma'} S(\rho||\sigma') = -\mathrm{tr}(\rho \log(\sigma)) + \mathrm{tr}(\rho \log(\rho)) \tag{2.6}$$

It is the distance from a state to its closest separable state σ. A visualisation of this quantity is illustrated in FIG. (2.1) bellow.

2.2.3 Negativity and Logarithmic Negativity

A matrix obtained from a partial transposition of a density matrix of a separable state will have positive eigenvalues [22]. The corresponding density matrix is then said to be PPT. If a density matrix has negative eigenvalues then it is NPT. Negative eigenvalues would imply non-separability. The absolute value of the sum

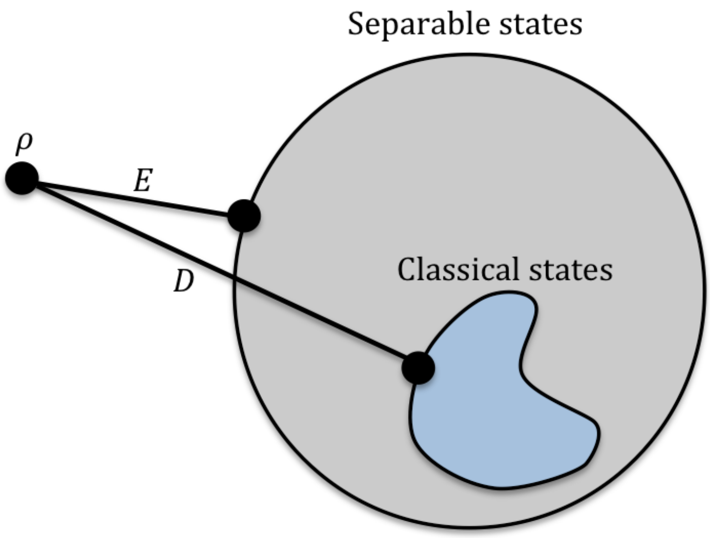

FIGURE 2.1: Visualisation of Entanglement and Discord in relative entropy measure

of these negative eigenvalues is known as *negativity* and it *measures entanglement for general states* [23, 24]. It is defined as follows

$$\mathcal{N}_{A-B} = \frac{||\rho^{T_A}||_1 - 1}{2} \tag{2.7}$$

where $||\rho^{T_A}||_1$ is trace norm of the density matrix after partial transposition with respect to A [24]. $||\rho^{T_A}||_1$ is simply $\sum_i |\lambda_i|$, where λ_is are eigenvalues of the density matrix after the partial transposition.

In tripartite systems, entanglement in the partition $i-jk$ is quantified by negativity in the following way

$$\mathcal{N}_{i-jk} = \frac{||\rho^{T_i}||_1 - 1}{2} \tag{2.8}$$

where $\{i, j, k\}$ is any permutation of $\{A, B, C\}$. Negativity in the partition $i - j$ is obtained by tracing out k. The value will obey $\mathcal{N}_{i-j} \leq \mathcal{N}_{i-jk}$ since tracing out k is just a local operation, so it cannot increase entanglement [24].

The significant advantage of using negativity as entanglement measure is that it is easy to compute. It does not require minimisation in the computation that is usually present in other measures. However it has a disadvantage that it does not guarantee separability if the negativity is zero. There are entangled states that are PPT [3]. In pure states, we do not have this disadvantage as will be shown bellow.

Compared to entropy of entanglement, negativity of Bell states equal to 0.5. In three-qubit pure states Schmidt decomposition always allow us to write the state of the system as

$$|\Psi\rangle = \sqrt{p_i}\,|0\rangle_i\,|\Psi_0\rangle_{jk} + \sqrt{1-p_i}\,|1\rangle_i\,|\Psi_1\rangle_{jk} \tag{2.9}$$

where $\langle\Psi_0\,|\,\Psi_1\rangle_{jk} = 0$ and $\{i, j, k\}$ is any permutation of $\{A, B, C\}$ [25]. The negativity $\mathcal{N}_{i-jk}$ is given by $\sqrt{p_i(1-p_i)}$. We will use this result in Chapter 4. It should also be noted that maximum value of negativity is dimension dependent. For example in $3 \times 2 \times 2$ dimension negativity in the partition $A - BC$ can reach as high as 1.

Logarithmic negativity is a measure of entanglement that stems from Negativity. It is defined as

$$\mathcal{L}_{A-B} = \log ||\rho^{T_A}||_1 = \log\left(2\,\mathcal{N}_{A-B} + 1\right) \tag{2.10}$$

While both negativity and logarithmic negativity are used to measure entanglement, they have different significance. The former bounds the teleportation capacity while the latter bounds the distillable entanglement of mixed states [24].

In pure states, distillable entanglement reduces to entropy of entanglement [3]. Then we have $\mathcal{L} \geq S_N$. If negativity equals zero, the logarithmic negativity also equals zero, then S_N must equal zero. Zero S_N implies no entanglement. So zero negativity implies zero entanglement.

2.2.4 Discord Measure

The original paper has defined a discord measure based on the difference between two expressions of mutual information that are classically equal [16]. Discord measure can also be defined from relative entropy as can be visualised in FIG. 2.1; it is the distance from a state to its closest classical state. However the *one way deficit* measure [18] will be used in this thesis as it has been used in deriving the bound on entanglement gain for general states [10, 11] . It is defined as

$$D_{A|B} = \min_{\{\Pi_B^i\}} S\left(\rho_{AB}||\sum_i \Pi_B^i \rho_{AB} \Pi_B^i\right) \tag{2.11}$$

where $S(X||Y) \equiv -\text{tr}(X \log Y) + \text{tr}(X \log X)$ and Πs are projectors.

Chapter 3

Entanglement Distribution Methods and Protocols

3.1 Methods

This Section reviews entanglement distribution between two separate systems with the help of an ancillary system. The setup can be seen in FIG. 1.1. Alice and Bob each has one system, A and B respectively, in their laboratories. C is the ancillary system that will be used to distribute entanglement. First, the case where C interacts discretely with A and B is reviewed in Subsection 3.1.1. Subsection 3.1.2 will then review the case where C continuously interacts with A and B along with Cubitt's argument that entanglement gain is not possible in pure states with separable C. Entanglement protocols that are useful in the discrete interaction case will be introduced in Section 3.2.

3.1.1 Discrete Interaction

In the discrete interaction case, as can be visualised in FIG. 3.1, C interacts only discretely (in a short time) with A and B. First, Alice makes A and C interact. Then C is sent to Bob. Finally, Bob makes B and C interact. Surprisingly, to entangle A and B, C can be separable at all times [8]. In other words, it does not require entanglement to create entanglement. The same procedure has also been applied recently by Kay [9]. It turns out that entanglement gain is bounded by a non-classical correlation in the partition $C - AB$ given by quantum discord [10, 11]. This bound is written as follows

$$E_{A-BC} - E_{B-AC} \leq D_{AB|C} \tag{3.1}$$

where $D_{AB|C}$ is the one way deficit. Recently, this bound has been experimentally tested [12, 13]. In pure states, discord measures entanglement. In Chapter 4 we will show that entanglement gain is bounded by entanglement in the partition $C - AB$. However violations of the bound are also found in some measures if system A has sufficiently high dimension.

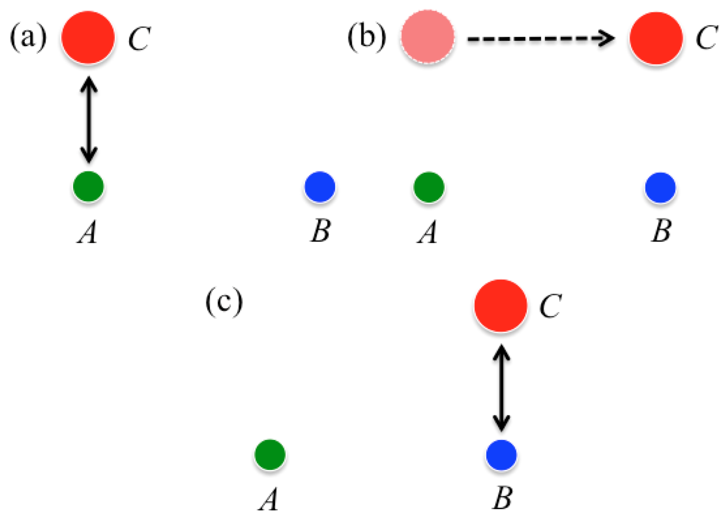

FIGURE 3.1: Entanglement distribution with discrete interaction. (a) C is made to interact with A in Alice's lab. (b) C is sent to Bob's lab. (c) C is made to interact with B

3.1.2 Continuous Interaction

Cubitt *et al.* [8] also mentioned a continuous interaction method; it can be visualised in FIG. 3.2. In this method, C is allowed to interact continuously with A and B. Cubit *et al.* argued that it is not possible to distribute entanglement with pure states if C is separable. The argument goes as follows: take $|a\rangle|b\rangle|c\rangle$ as an initial state. The state at small time Δt is given by

$$\begin{aligned} |\Psi(\Delta t)\rangle &= \left(\mathbb{1} - i\frac{(H_{AC}+H_{BC})}{\hbar}\Delta t\right)|a\rangle|b\rangle|c\rangle \\ &= (|a\rangle|b\rangle + \Delta t\,|\psi_{AB}\rangle)(|c\rangle + \Delta t\,|\psi_C\rangle) \end{aligned} \tag{3.2}$$

where $H_{AC}+H_{BC}$ is the interaction Hamiltonian and the separability condition of C has been used in the last equation. The state $|\Psi(\Delta t)\rangle$ is entangled in the partition A–B if $|\psi_{AB}\rangle$ has $|a_\perp\rangle|b_\perp\rangle$ component. It is clear from the first line of Eq. 3.2 that applying $\langle a_\perp|\langle b_\perp|$ to $|\Psi(\Delta t)\rangle$ would result in zero. Hence entanglement in the partition $A-B$ is zero. However, if we start with $|0\rangle|0\rangle|c\rangle$ state and after Δt we have $(|0\rangle|0\rangle + \Delta t\,|0\rangle|1\rangle + \Delta t\,|1\rangle|0\rangle)|c\rangle'$, the argument does not apply since there is entanglement in $|\Psi(\Delta t)\rangle$ despite having zero inner product with $\langle 1|\langle 1|$. The question is then whether there exists such Hamiltonian that will allow the state $|0\rangle|0\rangle|c\rangle$ to evolve to $(|0\rangle|0\rangle + \Delta t\,|0\rangle|1\rangle + \Delta t\,|1\rangle|0\rangle)|c\rangle'$. In Chapter 5, we will give a different way of proving that it is indeed not possible to distribute entanglement with pure states while C is separable.

Cubitt *et al.* also stated that it is possible to distribute entanglement if the state is mixed and C is separable. However, perturbation theory is used in the argument. Perturbation theory can create an artefact that entanglement can be distributed with pure states while C is separable as will be shown in Chapter 6. This artefact is in contradiction with their first argument about the use of pure states as discussed above. So Cubitt's argument about the use of mixed states should be considered with care. It is however impossible to distribute entanglement with mixed states and having C in a product form as will be proven in Chapter 5.

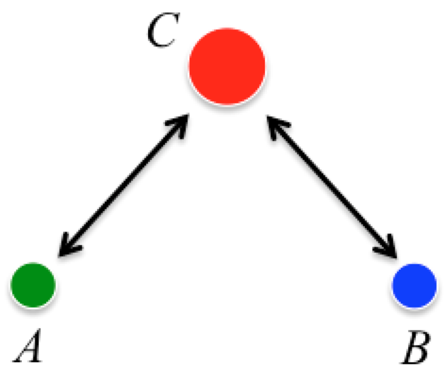

FIGURE 3.2: Entanglement distribution with continuous interaction. C is allowed to interact with A and B continuously.

3.2 Protocols

We shall interpret Eq. 3.1 as follows: entanglement gain is entanglement after the transfer of C minus the one before the transfer, that is $\Delta E \equiv E_{A-BC} - E_{B-AC}$. $D_{AB|C}$ is what is needed or communicated in the process to entangle A and B. In pure states we shall use E_{C-AB} instead of $D_{AB|C}$. We can then classify entanglement distribution into *excessive* and *non-excessive*. Excessive protocol is when entanglement gain is higher than entanglement communicated while non-excessive protocol is the other way around. These protocols can be written as follows

$$\begin{aligned} \Delta E \leq \quad & E_{\text{com}} \quad \text{non-excessive} \\ \Delta E > \quad & E_{\text{com}} \quad \text{excessive} \end{aligned}$$

Examples of non-excessive protocol would be the direct method in which entanglement is first created in one lab and the indirect method in which entanglement swapping is used as mentioned in Chapter 1. Excessive protocol is obviously preferable since it does not require as much entanglement in the partition $C - AB$ as what is gained. In mixed states, the entanglement communicated can be zero as it is possible to have non-zero discord in separable states. Hence the R.H.S of Eq. 3.1 can be non-zero and the protocol is excessive. In Chapter 4, we shall prove that excessive entanglement protocol is also possible in pure states.

Chapter 4

Entanglement Distribution with Discrete Interaction

4.1 Pure States

This Section presents proofs that entanglement inequality given in Eq. (4.1) bellow holds for some measures and dimensions in pure states. Note that this inequality will lead to non-excessive distribution as we replace i, j, and k with A, B, and C respectively. First, we will look at entropy of entanglement and linear entropy measures in 4.1.1. Then 4.1.2 and 4.1.3 will present the proof for negativity and logarithmic negativity measures respectively. Violations of the inequality, leading to excessive distribution, are also presented in these Subsections.

$$E_{i-jk} - E_{j-ik} \leq E_{k-ij} \tag{4.1}$$

where $\{i, j, k\}$ is any permutation of $\{A, B, C\}$.

4.1.1 Entropy of Entanglement and Linear Entropy

Theorem 4.1. *von Neumann and linear entropy are non-excessive*

Proof. Properties of some measure:

1. Subadditivity: $M_{ij} \leq M_i + M_j$
2. $M_{ij} = M_k$
3. M_i quantifies entanglement in the partition $i - jk$

If a measure satisfies the set of properties above then it will follow that Eq. (4.1) holds for such measure.

Both von Neumann entropy and linear entropy satisfy properties given above [26, 27]. Thus, the entanglement inequality holds in both measures, leading to non-excessive distribution. □

4.1.2 Negativity

Theorem 4.2. *Negativity is non-excessive in a system of three qubits*

Proof. Since linear entropy satisfies property 1 and 2 stated in the previous Subsection we will have

$$1 - \mathrm{tr}(\rho_k^2) \quad \leq \quad 1 - \mathrm{tr}(\rho_i^2) + 1 - \mathrm{tr}(\rho_j^2) \tag{4.2}$$

Recall that in this system we have Schmidt decomposition: $|\Psi\rangle = \sqrt{p_i}\,|0\rangle_i\,|\Psi_0\rangle_{jk} + \sqrt{1-p_i}\,|1\rangle_i\,|\Psi_1\rangle_{jk}$ and negativity is given by $\mathcal{N}_{i-jk} = \sqrt{p_i(1-p_i)}$. So Eq. (4.2) becomes

$$\begin{aligned} 2p_k(1-p_k) &\leq 2p_i(1-p_i)+2p_j(1-p_j) \\ \mathcal{N}^2_{k-ij}-\mathcal{N}^2_{j-ik} &\leq \mathcal{N}^2_{i-jk} \end{aligned} \tag{4.3}$$

Let us consider two cases: 1. $\mathcal{N}_{k-ij} \geq \mathcal{N}_{j-ik}$. 2. $\mathcal{N}_{k-ij} < \mathcal{N}_{j-ik}$. In the first case we have:

$$(\mathcal{N}_{k-ij}-\mathcal{N}_{j-ik})^2 \leq \mathcal{N}^2_{k-ij}-\mathcal{N}^2_{j-ik} \leq \mathcal{N}^2_{i-jk} \tag{4.4}$$

Taking the square root will give us:

$$\mathcal{N}_{k-ij}-\mathcal{N}_{j-ik} \leq \mathcal{N}_{i-jk} \tag{4.5}$$

which is essentially Eq. (4.1) after a permutation. In the second case the L.H.S of Eq. (4.5) will be negative while the R.H.S is positive. So Eq. (4.5) still holds. □

For higher dimensions, for example in $3\times2\times2$ system, a counter example is given by $|\Psi\rangle = 1/\sqrt{3}(|001\rangle+|110\rangle+|200\rangle)$. In this state, $\mathcal{N}_{A-BC}-\mathcal{N}_{B-AC} = 1-\sqrt{2}/3 \approx 0.529$ while $\mathcal{N}_{C-AB} = \sqrt{2}/3 \approx 0.471$. So negativity inequality does not hold thus the distribution is excessive. This is due to the fact that the dimension of A is higher thus $\mathcal{N}_{A-BC}$ is favoured. If $\mathcal{N}_{B-AC}$ or $\mathcal{N}_{C-AB}$ is favoured there will be no violation in this Hilbert space. So if Eq. (4.1) holds in $2\times2\times2$ dimension it should hold in Hilbert space of arbitrary dimension as long as A has a dimension of 2. On the other hand, if the dimension of A is higher than 2 the corresponding Hilbert space will contain the counter example given above, allowing for excessive distribution.

4.1.3 Logarithmic Negativity

Theorem 4.3. *Logarithmic negativity is non-excessive in a system of three qubits*

Proof. The proof follows from Theorem 4.2. In a system of three qubits we have negativity inequality, which after some manipulations gives us

$$\log\left(2\,\mathcal{N}_{j-ik} + 2\,\mathcal{N}_{k-ij} + 1\right) \geq \log\left(2\,\mathcal{N}_{i-jk} + 1\right) \tag{4.6}$$

Since $\log\left(2\,\mathcal{N}_{j-ik} + 1\right) + \log\left(2\,\mathcal{N}_{k-ij} + 1\right) \geq \log\left(2\,\mathcal{N}_{j-ik} + 2\,\mathcal{N}_{k-ij} + 1\right)$ we will have $\log\left(2\,\mathcal{N}_{j-ik} + 1\right) + \log\left(2\,\mathcal{N}_{k-ij} + 1\right) \geq \log\left(2\,\mathcal{N}_{i-jk} + 1\right)$, which is essentially

$$\mathcal{L}_{i-jk} - \mathcal{L}_{j-ik} \leq \mathcal{L}_{k-ij} \tag{4.7}$$

□

The proof above also means that negativity inequality implies logarithmic negativity inequality. In higher dimensions where there are violations of negativity inequality it gets slightly more complicated. Bellow I will present numerical evidence that Eq. (4.7) still holds in $3\times2\times2$ dimension and conjecture that it should also hold in Hilbert space of arbitrary dimension as long as the dimension of A is less than 4.

Let us define the violation as $v \equiv \mathcal{N}_{i-jk} - (\mathcal{N}_{j-ik} + \mathcal{N}_{k-ij})$. So the new inequality that will hold in this system would be $\mathcal{N}_{j-ik} + \mathcal{N}_{k-ij} + v \geq \mathcal{N}_{i-jk}$. After some manipulations it can be written as

$$\log\left(2\,\mathcal{N}_{j-ik} + 2\,\mathcal{N}_{k-ij} + 2v + 1\right) \geq \log\left(2\,\mathcal{N}_{i-jk} + 1\right) \tag{4.8}$$

The value of v depends on $\mathcal{N}_{j-ik}$ and $\mathcal{N}_{k-ij}$. An example of v distribution in $3\times2\times2$ system is given in FIG. (4.1). Numerical computation suggests that $\log\left(2\,\mathcal{N}_{j-ik} + 1\right) + \log\left(2\,\mathcal{N}_{k-ij} + 1\right) \geq \log\left(2\,\mathcal{N}_{j-ik} + 2\,\mathcal{N}_{k-ij} + 2v + 1\right)$ holds in $3\times2\times2$ dimension; Combining it with Eq. (4.8) will give Eq. 4.7.

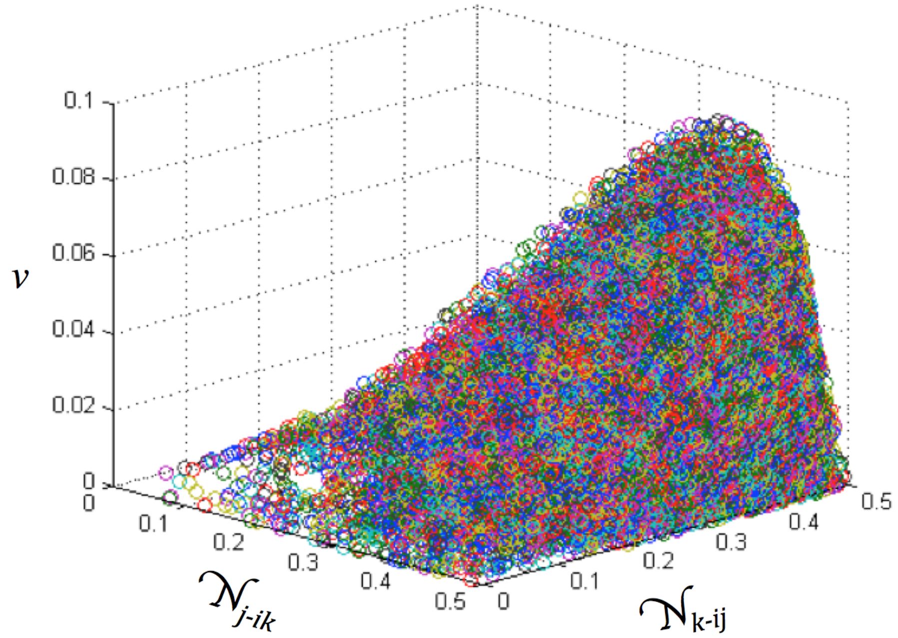

FIGURE 4.1: The distribution of v as a function of $\mathcal{N}_{j-ik}$ and $\mathcal{N}_{k-ij}$ in $3\times2\times2$ dimension

In $4 \times 2 \times 2$ system, a counter example is given by $|\Psi\rangle = 1/\sqrt{103}(10\,|000\rangle + |110\rangle + |201\rangle + |311\rangle)$. $\mathcal{L}_{A-BC} - \mathcal{L}_{B-AC} = \log\left(169/(2\sqrt{202}+103)\right) \approx 0.363$ while $\mathcal{L}_{C-AB} = \log\left((2\sqrt{202}+103)/103\right) \approx 0.352$. The gain is higher than what is communicated, which is excessive. Similar to the violations in negativity inequality it is caused by the dimension of A. If $\mathcal{L}_{B-AC}$ or $\mathcal{L}_{C-AB}$ is favoured the inequality will likely to hold. Therefore if Eq. (4.7) holds in $2\times2\times2$ and $3\times2\times2$ dimensions it should also hold in Hilbert space of arbitrary dimension as long as the dimension of A is 3 or lower. On the other hand, Hilbert space with arbitrary dimension with A has a dimension higher than 3 will contain the counter example given above hence allowing for excessive distribution.

4.2 Multiple Discrete Interactions

Here we shall derive entanglement bound for non-excessive distribution in pure states where discrete interaction method is applied a number of times to maximise entanglement gain. In this scenario C is "jumping" n times between A and B, see FIG. 4.2. Entanglement inequality should apply when C is being sent to Bob as well as when it is to Alice. Assume that C is at Alice's lab at the end. This method has been used before for general states [10].

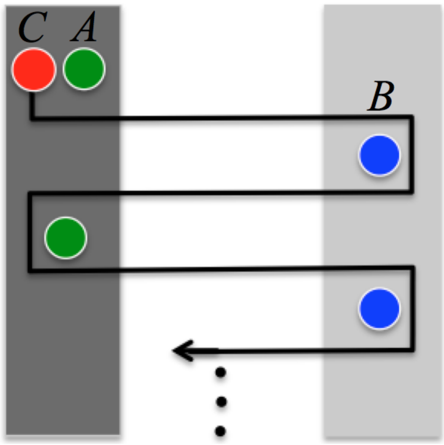

FIGURE 4.2: Entanglement distribution with multiple discrete interactions in pure states

Applying the entanglement inequality in the first "jump" from A to B will yield $E^1_{A-BC} - E^1_{B-AC} \leq E^1_{C-AB}$. After C interacts with B, the entanglement is given by E^2_{A-BC} and it is smaller than E^1_{A-BC} since the interaction is local in $A-BC$ partition. Hence the following inequality holds: $E^2_{A-BC} - E^1_{B-AC} \leq E^1_{C-AB}$. Similarly applying the same technique to other jumps will yield

$$\begin{aligned}
E^2_{A-BC} - E^1_{B-AC} &\leq E^1_{C-AB} \\
E^3_{B-AC} - E^2_{A-BC} &\leq E^2_{C-AB} \\
E^4_{A-BC} - E^3_{B-AC} &\leq E^3_{C-AB} \\
&\vdots \\
E^n_{A-BC} - E^{n-1}_{B-AC} &\leq E^{n-1}_{C-AB} \\
E^{n+1}_{B-AC} - E^n_{A-BC} &\leq E^n_{C-AB} \qquad (4.9)
\end{aligned}$$

Summing them up will result in $E^{n+1}_{B-AC} - E^{1}_{B-AC} \leq \sum_i E^{i}_{C-AB}$. Let us define the final entanglement as $E_{\text{final}} \equiv E^{n+1}_{B-AC}$ and initial entanglement as $E_{\text{initial}} \equiv E^{0}_{B-AC}$ that is before Alice interacts C with A. $E^{0}_{B-AC} \geq E^{1}_{B-AC}$ because the interaction is local in $B - AC$ partition. Hence we will have

$$E_{\text{final}} - E_{\text{initial}} \leq \sum_i E^{i}_{C-AB} \tag{4.10}$$

The overall entanglement grow in the partition $B - AC$ is bounded by the sum of entanglement in the partition $C - AB$. Similar technique can be applied to obtain a bound on entanglement grow in the partition $A - BC$.

Chapter 5

Entanglement Distribution with Continuous Interaction

This Chapter provides proofs that entanglement gain is not possible under some circumstances. First, pure states and separable ancillary system is considered in Section 5.1. Next, Section 5.2 will consider a special case: Born approximation in which the state of the system is evolving in way that the ancillary system stays constant and in a product form. We will also discuss how entanglement changes with interaction strength in weak interaction limit.

5.1 Pure States

Theorem 5.1. *Entanglement in the partition $A-B$ cannot grow if the state of C is separable from AB*

Proof. First we are going to use $|a\rangle |b\rangle |c\rangle$ as the initial state and show that entanglement cannot grow. Then we will prove the same if one starts with an entangled state in the partition $A - B$, i.e. we will use $(k_1 |a\rangle |b\rangle + k_2 |a_\perp\rangle |b_\perp\rangle) |c\rangle$ without loss of generality.

The evolution operator reads $U = \exp(-iH_{AC}t/\hbar - iH_{BC}t/\hbar)$. Let us use Trotter expansion (derivation is in the appendix), then the evolution operator can be written as

$$U = \lim_{n\to\infty} \left(e^{-i\frac{H_{AC}\Delta t}{\hbar}} e^{-i\frac{H_{BC}\Delta t}{\hbar}} \right)^n = \lim_{n\to\infty} \left(e^{-i\frac{H_{BC}\Delta t}{\hbar}} e^{-i\frac{H_{AC}\Delta t}{\hbar}} \right)^n \tag{5.1}$$

where $\Delta t = t/n$ and we have used the fact that H_{AC} and H_{BC} are exchangeable in the original form of the evolution operator. Δt is assumed to be very small such that during that time the state can be evolved by applying $\exp(-iH_{AC}\Delta t/\hbar)$ $\exp(-iH_{BC}\Delta t/\hbar)$ or $\exp(-iH_{BC}\Delta t/\hbar)\exp(-iH_{AC}\Delta t/\hbar)$.

Let us use $U_{AC} \equiv \exp(-iH_{AC}\Delta t/\hbar)$ and $U_{BC} \equiv \exp(-iH_{BC}\Delta t/\hbar)$. Then the evolution operator can be written as $U_{AC}\, U_{BC}$ or $U_{BC}\, U_{AC}$; both of which must yield the same result.

Now let us start with $|a\rangle |b\rangle |c\rangle$ and apply $U_{AC}\, U_{BC}$. The state will evolve as

$$U_{AC}\, U_{BC} |a\rangle |b\rangle |c\rangle = U_{AC} |a\rangle |\psi\rangle_{bc} = |\psi\rangle'_{ab} |c\rangle' \tag{5.2}$$

where we have used the constraint that C must be separable in the last step. Using Schmidt decomposition on the last equality, we will have the following property of U_{AC}:

$$U_{AC} |a\rangle \sum_j d_j |b\rangle^j |c\rangle^j = \sum_j d_j |a_j\rangle |b\rangle^j |c\rangle' \tag{5.3}$$

where $\{|a_j\rangle\}$ are orthonormal bases of A. The reason is as follows: The last step in Eq. (5.2) must be true for every state $|a\rangle |\psi\rangle_{bc}$. Therefore every term on the

L.H.S of Eq. (5.3) must factor out $|c\rangle'$.

Next apply $U_{BC}\, U_{AC}$ to the initial state, which will be written as $\sum_j d_j \, |a\rangle \, |b\rangle \, |j\rangle$. Based on the property of U_{AC} above, the state will evolve as

$$U_{BC} \sum_j d_j \, |a_j\rangle \, |b\rangle \, |c\rangle' = (\sum_j d_j \, |a_j\rangle) \, |\psi_{bc}\rangle' = (\sum_j d_j \, |a_j\rangle) \, |b\rangle' \, |c\rangle' = |a\rangle' \, |b\rangle' \, |c\rangle' \tag{5.4}$$

It is clear from eqn. 5.4 that entanglement in the partition $A - B$ remains zero.

When one starts with $(k_1 \, |a\rangle \, |b\rangle + k_2 \, |a_\perp\rangle \, |b_\perp\rangle) \, |c\rangle$, the property of U_{AC} in eqn. 5.3 can be used to the first and second terms since $|c\rangle'$ has to be factored out from each term. So, rewriting the state as $\sum_j d_j(k_1 \, |a\rangle \, |b\rangle + k_2 \, |a_\perp\rangle \, |b_\perp\rangle) \, |j\rangle$ and applying $U_{BC}\, U_{AC}$ would result in

$$\begin{aligned} & U_{BC} \sum_j (k_1(d_j \, |a_j\rangle) \, |b\rangle + k_2(d_j \, |a_{\perp j}\rangle) \, |b_\perp\rangle) \, |c\rangle' \\ & = \sum_j \left(k_1(d_j \, |a_j\rangle) \, |b\rangle' + k_2(d_j \, |a_{\perp j}\rangle) \, |b_\perp\rangle'\right) |c\rangle' \\ & = \left(k_1 \, |a\rangle' \, |b\rangle' + k_2 \, |a_\perp\rangle' \, |b_\perp\rangle'\right) |c\rangle' \end{aligned} \tag{5.5}$$

$|a\rangle'$ and $|a_\perp\rangle'$ are orthogonal to each other because the application of U_{AC} preserves the orthogonality. The same is true for $|b\rangle'$ and $|b_\perp\rangle'$ after the application of U_{BC}. The coefficients remain the same with the ones in the initial state. Hence, entanglement in the partition $A - B$ at any time t is the same as the initial entanglement. In other words, entanglement cannot grow. □

Theorem 5.1 states that if the states are pure and C is separable entanglement in the partition $A - B$ cannot grow. This implies that if E_{A-B} grows, there has to be entanglement in the partition $C - AB$. Then there is a question whether entanglement in the partition $C - AB$ bounds the entanglement grow.

5.2 Special Case: Born Approximation

In this Section we will prove that entanglement in the partition $A-BC$ cannot grow if the system evolves as in the Born approximation. Entanglement measure used in this proof is the relative entropy. In this approximation the evolution of the state reads

$$\rho_{ABC}(0) = \rho_{AB}(0) \otimes \rho_C \rightarrow \rho_{ABC}(t) = \rho_{AB}(t) \otimes \rho_C \tag{5.6}$$

where we have assumed that the initial state is of the form $\rho_{AB}(0) \otimes \rho_C$. We will begin by proving three lemmas and use them to prove the theorem. Next we will discuss entanglement behaviour with respect to interaction strength in the weak interaction limit.

Lemma 5.2. *$E_{A-BC}(t) \leq E_{A-BC}(0)$ if $U\sigma_{A-BC}(0)U^\dagger$ is separable in the partition $A-BC$, where $\sigma_{A-BC}(0)$ is the closet separable state to $\rho_{ABC}(0)$*

Proof. At time $t = 0$, entanglement in the partition $A-BC$ is given by

$$\begin{aligned} E_{A-BC}(0) &= S\left(\rho_{ABC}(0)||\sigma_{A-BC}(0)\right) \\ &= -\mathrm{tr}\left(\rho_{ABC}(0)\log\sigma_{A-BC}(0)\right) + \mathrm{tr}\left(\rho_{ABC}(0)\log\rho_{ABC}(0)\right) \end{aligned} \tag{5.7}$$

where $E_{A-BC}(0)$ is minimised, i.e. $\sigma_{A-BC}(0)$ is the closest separable state to $\rho_{ABC}(0)$. From the first term in Eq. (5.7) and using the cyclic property of trace: $-\mathrm{tr}\left(U\rho_{ABC}(0)U^\dagger U\log\sigma_{ABC}(0)U^\dagger\right)$. Then $U\log\sigma_{ABC}(0)U^\dagger =$ $\sum_k U\log p_k\,|k\rangle\langle k|\,U^\dagger = \sum_k \log p_k\,|\psi_k\rangle\langle\psi_k| = \log U\sigma_{ABC}(0)U^\dagger$ where we have written $\sigma_{ABC}(0)$ as $\sum_k p_k\,|k\rangle\langle k|$ and $|\psi_k\rangle = U\,|k\rangle$ is the basis after the evolution. So we are left with $-\mathrm{tr}\left(U\rho_{ABC}(0)U^\dagger\log U\sigma_{ABC}(0)U^\dagger\right)$. Apply the same technique to the second term in Eq. (5.7) we will get:

$$E_{A-BC}(0) = S\left(\rho_{ABC}(t)||U\sigma_{A-BC}(0)U^\dagger\right) \tag{5.8}$$

On the other hand, entanglement at time t is

$$E_{A-BC}(t) = \min_{\sigma'_{A-BC}} S\left(\rho_{ABC}(t)||\sigma'_{A-BC}\right) \tag{5.9}$$

If $U\sigma_{A-BC}U^{\dagger}$ is separable is the partition $A-BC$, that means $E_{A-BC}(t) \leq E_{A-BC}(0)$ as can be visualised in Fig. 5.1.

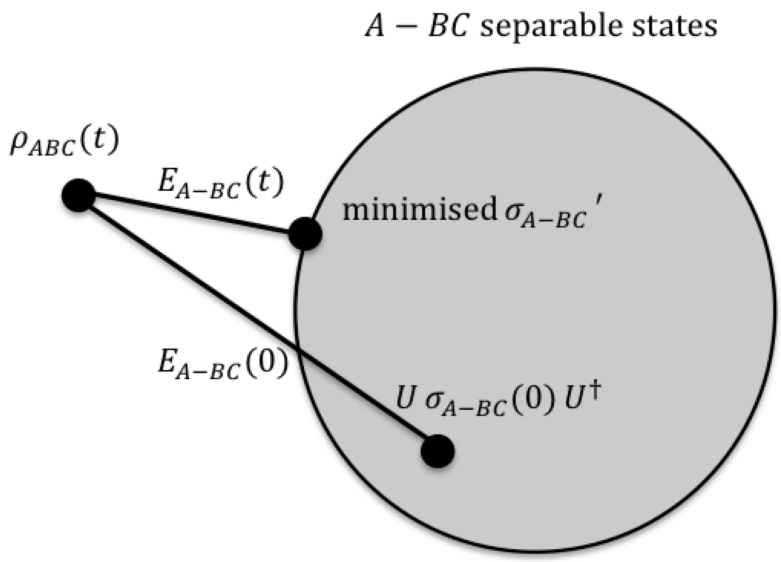

FIGURE 5.1: Visualisation of $E_{A-BC}(0)$ and $E_{A-BC}(t)$

□

Lemma 5.3. *$\sigma_{A-BC}(0)$ is of the form $\sigma_{AB}(0) \otimes \rho_C$ where $\sigma_{AB}(0)$ is the closest separable state to $\rho_{AB}(0)$.*

Proof. Let the closest separable state to $\rho_{ABC}(0)$ is of the form $\sigma_{AB}(0) \otimes \rho_C$. If X_{A-BC} is a closer separable state, it will follow that

$$S\left(\rho_{ABC}(0)||\sigma_{AB}(0) \otimes \rho_C\right) - S\left(\rho_{ABC}(0)||X_{A-BC}\right) \geq 0 \tag{5.10}$$

Using the additivity of log and linearity of trace we have the identity: $\mathrm{tr}(\rho \log(\Pi_1 \otimes \Pi_2)) = \mathrm{tr}(\mathrm{tr}_2(\rho)\log \Pi_1) + \mathrm{tr}(\mathrm{tr}_1(\rho)\log \Pi_2)$. Applying the identity to the first term in (5.10): $S(\rho_{ABC}(0)||\sigma_{AB}(0)\otimes\rho_C) = S(\rho_{AB}(0)||\sigma_{AB}(0)) = E_{A-B}(0)$. So that (5.10) becomes

$$E_{A-B}(0) - S(\rho_{ABC}(0)||X_{A-BC}) \geq 0 \tag{5.11}$$

Use the fact that $E_{A-B} \leq E_{A-BC}$ since tracing out C is a local operation. Then we have

$$E_{A-BC}(0) - S(\rho_{ABC}(0)||X_{A-BC}) \geq E_{A-B}(0) - S(\rho_{ABC}(0)||X_{A-BC}) \geq 0 \tag{5.12}$$

Since X is separable in the partition $A-BC$ and $E_{A-BC}(0)$ is minimised, the first term in (5.12) can only be smaller than zero. It follows that equality is the only solution in (5.10) and hence X_{A-BC} must equal to $\sigma_{AB}(0)\otimes\rho_C$.

□

Lemma 5.4. *$U\sigma_{AB}(0)\otimes\rho_C U^\dagger$ is separable in the partition $A-BC$.*

Proof. We are going to use the following infinitesimal evolution operator $U(\Delta t) = \mathbb{1} - i(H_{AC}+H_{BC})\Delta t/\hbar$ and write $\sigma_{AB}(0)$ as $\sum_i p_i\, \Pi_A^i \otimes \Pi_B^i$. The evolution then reads

$$U(\Delta t)\sigma_{AB}(0)\otimes\rho_C U^\dagger(\Delta t) =$$
$$\left(\mathbb{1} - i\frac{\Delta t}{\hbar}H\right)\left(\sum_i p_i\, \Pi_A^i \otimes \Pi_B^i \otimes \rho_C\right)\left(\mathbb{1} + i\frac{\Delta t}{\hbar}H\right) \tag{5.13}$$

It follows from the evolution of state in the Born approximation that $\sigma_{AB}(0)\otimes\rho_C$ must evolve to $\sigma_{AB}(t)\otimes\rho_C$ since (5.6) must be true for any initial state $\rho_{AB}(0)\otimes\rho_C$. In this evolution, mutual information $I(AB-C) = I(A-C) = I(B-C) = 0$

because C has no correlation with any other system. Expanding Eq. (5.13):

$$\sum_i p_i \, (\Pi_A^i \otimes \Pi_B^i \otimes \rho_C - i\frac{\Delta t}{\hbar} H_{AC}\Pi_A^i \otimes \Pi_B^i \otimes \rho_C - i\frac{\Delta t}{\hbar} H_{BC}\Pi_A^i \otimes \Pi_B^i \otimes \rho_C$$

$$+ i\frac{\Delta t}{\hbar}\Pi_A^i \otimes \Pi_B^i \otimes \rho_C H_{AC} + i\frac{\Delta t}{\hbar}\Pi_A^i \otimes \Pi_B^i \otimes \rho_C H_{BC}) \tag{5.14}$$

where higher order terms of the interaction Hamiltonian have been ignored since they are very small. The assumption in the evolution of the separable state must be true for every independent component of the Hamiltonian. For example if we vary H_{AC}, every term in Eq. (5.14) that has H_{AC} must add up and ρ_C has to be factored out as a product state. From mutual information theory we have: $I(X-Y|Z) = I(X-YZ) - I(X-Z) = I(Y-X|Z) = I(Y-XZ) - I(Y-Z)$ [28]. Apply it to our system, i.e. replace X, Y, and Z with A, B, and C respectively:

$$I(A-BC) - I(A-C) = I(B-AC) - I(B-C) \tag{5.15}$$

Every term in Eq. 5.14 that has H_{AC} will have $I(B-AC) = 0$. $I(A-BC)$ will also be zero as a consequence of Eq. 5.15. Hence the terms will add up and result in a product state. The same technique can be applied to terms with H_{BC}. Therefore, Eq. 5.14 can be written as $\sum_i k_i \, {\Pi_A^i}' \otimes {\Pi_B^i}' \otimes \rho_C$ and it is separable in the partition $A-BC$. Consequent applications of small unitary operator will also yield separable state in the partition $A-BC$ hence proving the lemma. □

Theorem 5.5. *In the Born approximation, entanglement in the partition $A-BC$ cannot grow*

Proof. By lemma 5.3 and 5.4 we know that $U\sigma_{A-BC}U^\dagger$ is separable in the partition $A-BC$; this together with lemma 5.2 gives the theorem. □

In reality however, the state of C will not be constant in weak interaction limit. In what follows, we will assume that the state evolves as

$$\rho_{ABC}(0) = \rho_{AB}(0) \otimes \rho_C \rightarrow \rho_{ABC}(t) = (1-\epsilon)\rho_{AB}(t) \otimes \rho_C + \epsilon \rho'_{ABC}(t) \tag{5.16}$$

Entanglement measures that are convex can be used to find a bound on entanglement at time t. Convexity is given by

$$E(\sum_i p_i\, \rho_i) \leq \sum_i p_i\, E(\rho_i) \tag{5.17}$$

Applying the convex function to Eq. 5.16 will yield

$$E(\rho_{ABC}(t)) \leq (1-\epsilon)E(\rho_{AB}(t) \otimes \rho_C) + \epsilon E(\rho'_{ABC}(t)) \tag{5.18}$$

If one starts with zero entanglement in the partition $A-BC$ then $E(\rho_{AB}(t) \otimes \rho_C) = 0$ is implied from Theorem 5.5. So the following bound holds in the partition $A - BC$

$$E(\rho_{ABC}(t)) \leq \epsilon\, E(\rho'_{ABC}(t)) \tag{5.19}$$

The weaker the interaction the smaller the entanglement. This bound will be tested in Chapter 6 where negativity (which is convex [24]) is used as entanglement measure.

Chapter 6

Evolution of Entanglement

This chapter contains computation on entanglement as a function of time. For the ease of computation, dimensionless quantities have been used: H is dimensionless Hamiltonian and $x \equiv \omega t$ is dimensionless time. Section 6.1 presents entanglement calculation using perturbation theory on the Hamiltonian as was used by Cubitt *et al.* [8]. The problem that arises from such method will also be discussed. Section 6.2 presents entanglement without the use of perturbation theory. We will investigate entanglement behaviour in weak and strong interaction limit.

6.1 Pure State with Perturbation

Motivated by ion-trap experiments [14] and cavity-QED experiments [14], an example of Hamiltonian was used by Cubitt *et al.* [8]. It is given by:

$$H = \mathbb{1}_{AB} \otimes \hat{c}^\dagger \hat{c} + \frac{\epsilon}{2}(\sigma_A^x \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \sigma_B^x) \otimes (\hat{c} + \hat{c}^\dagger) \tag{6.1}$$

where $\hat{c} = |0\rangle\langle 1| + \sqrt{2}\,|1\rangle\langle 2|$, $\hat{c}^\dagger$ is the adjoint of $\hat{c}$, σ_A^x and σ_B^x are the $\hat{x}$ pauli matrices of subsystems A and B respectively, and ϵ is a small parameter characterizing the strength of the interaction.

Rearranging the Hamiltonian gives:

$$H = \Pi_{++}^{AB} \otimes H_+^C + \Pi_{--}^{AB} \otimes H_-^C + (\Pi_{+-}^{AB} + \Pi_{-+}^{AB}) \otimes H_0^C \tag{6.2}$$

where $H_\pm^C = H_0 \pm \epsilon\,(\hat{c} + \hat{c}^\dagger)$, $H_0 = \hat{c}^\dagger\hat{c}$, and the Πs are projectors of systems A and B. Using perturbation theory, $H_\pm^C \approx \mathrm{diag}(-\epsilon^2, 1-\epsilon^2, 2+2\epsilon^2) \equiv H_p$, accurate to third order in ϵ. Then the evolution operator can be written as: $U = U_{\text{eff}} + O(\epsilon^4 t) + O(\epsilon)$, where $U_{\text{eff}} = \exp(-iH_{\text{eff}}\,\omega t)$ is effective evolution operator after neglecting higher order terms in the Hamiltonian and $H_{\text{eff}} = (\Pi_{++}^{AB} + \Pi_{--}^{AB}) \otimes H_p + (\Pi_{+-}^{AB} + \Pi_{-+}^{AB}) \otimes H_0^C$ is dimensionless effective Hamiltonian. The $O(\epsilon^4 t)$ term comes from the perturbation of the eigenvalues and the $O(\epsilon)$ term comes from the perturbation of the eigenstates. Cubitt *et al.* [8] pointed out that under U_{eff}, the $|{++}\rangle$ and $|{--}\rangle$ states acquire phase difference $\exp(-i2\epsilon^2\omega t)$ relative to the $|{+-}\rangle$ and $|{+-}\rangle$ states if C is in the $|2\rangle$ state and $\exp(i\epsilon^2\omega t)$ if C is in the $|0\rangle$ or $|1\rangle$ state.

Using the same U_{eff} as mentioned above, we try to evolve $|000\rangle$ state which is a pure product state and has no correlation. Applying the effective evolution operator on the $|000\rangle$ state gives:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle \otimes |b_1\rangle + |-\rangle \otimes |b_1\rangle\right] \otimes |0\rangle \tag{6.3}$$

where $|b_1\rangle = (\exp(i\epsilon^2\omega t)\,|+\rangle + |-\rangle)/\sqrt{2}$ and $|b_2\rangle = (\exp(i\epsilon^2\omega t)\,|-\rangle + |+\rangle)/\sqrt{2}$. The inner product $\langle b_1 \,|\, b_2\rangle$ is equal to $\cos(\epsilon^2\omega t)$ which is equal to zero when $\epsilon^2\omega t = \pi/2 + n\pi$ where n is a positive integer. Hence $|\Psi(t)\rangle$ can be entangled in the partition $A - B$.

Entropy of entanglement of the above state is given by:

$$\begin{aligned} E_{A-B}(t) &= \frac{1+\cos\epsilon^2\omega t}{2}\log\frac{2}{1+\cos\epsilon^2\omega t} \\ &+\frac{1-\cos\epsilon^2\omega t}{2}\log\frac{2}{1-\cos\epsilon^2\omega t} \end{aligned} \tag{6.4}$$

This function is plotted in Fig. 6.1.

Systems A and B can be maximally entangled. However, there appears to be no correlation in the partition $C-AB$ as can be seen from Eq. (6.3). This is in contradiction with the theory derived in Section 5. Therefore perturbation theory creates an artefact that entanglement gain is possible with pure sates and separable C. This artefact stems from entanglement in the partition $C-AB$, which is not exactly zero since there is an error in the state leading to an error in E_{C-AB}. This correlation is then responsible for the distribution of entanglement in the partition $A-B$. Another way of looking at this is to apply $\langle 110|$ to the state at small time Δt. The result is small but non-zero: $i\epsilon^2\omega\Delta t$, which according to the argument presented in Subsection 3.1.2 means there is H_{AB} component in the effective Hamiltonian. Perturbation theory was also used by Cubitt *et al.* to prove that it is possible to achieve entanglement gain using mixed states with separable C. This should be carefully verified as perturbation theory can result in similar artefact.

6.2 Without Perturbation

In this Section we shall investigate entanglement behaviour with respect to interaction strength in both weak and strong interaction limit. Computation was done using Hamiltonian given by Eq. (6.1) with the addition of free Hamiltonian for

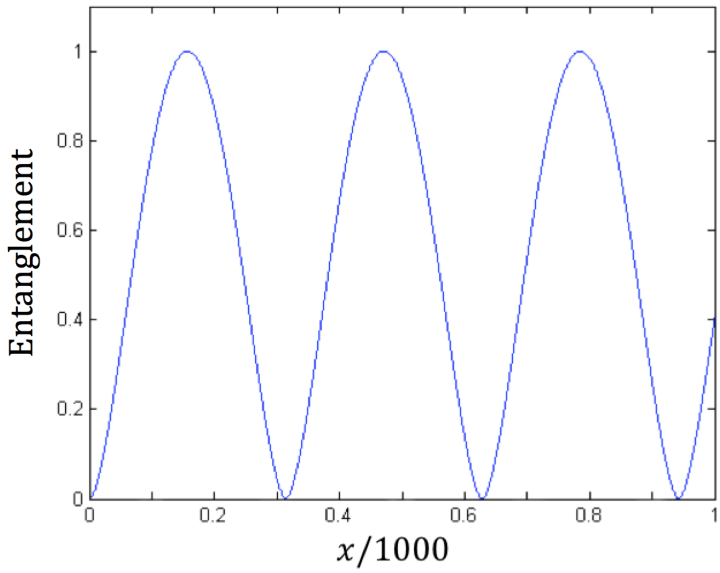

FIGURE 6.1: Entropy of entanglement in the partition $A-B$ as a function of dimensionless time with perturbation theory

system A and B, that is

$$H = \sigma_A^z \otimes \mathbb{1}_B \otimes \mathbb{1}_C + \mathbb{1}_A \otimes \sigma_B^z \otimes \mathbb{1}_C + \mathbb{1}_{AB} \otimes \hat{c}^\dagger \hat{c} + \frac{\epsilon}{2}(\sigma_A^x \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \sigma_B^x) \otimes (\hat{c} + \hat{c}^\dagger) \quad (6.5)$$

And $|000\rangle\langle 000|$ is used as the initial state.

6.2.1 Weak Interaction

FIG. 6.2 shows entanglement as a function of x. The interaction strength is varied: $\epsilon = 0.1, 0.01, 0.001$. Entanglement in the partition $A-BC$ is reduced as ϵ decreases. This is in agreement with what was predicted in weak interaction limit as stated in Chapter 5. A comparison on negativity amplitude for different interaction strength is given in TABLE. 6.1. It shows that the negativity amplitude is proportional to the interaction strength. The proportionality is better in small ϵ, it is shown by $\mathcal{N}_{\max}/\epsilon$ that is approaching a constant value. This is because the smaller the interaction strength the closer the state evolution to the one stated in Eq. (5.16). FIG. 6.2 also shows that maximum entanglement is achieved at the same time for different values of ϵ. A contrast behaviour will be shown in strong interaction limit.

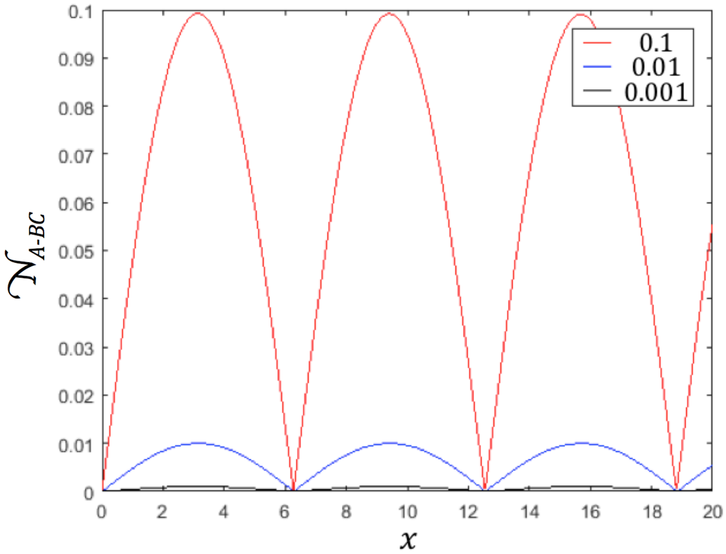

FIGURE 6.2: Negativity in the partition $A - BC$ as a function of dimensionless time for different values of interaction strength $\epsilon = 0.1, 0.01, 0.001$

TABLE 6.1: The ratio of negativity amplitude in the partition $A - BC$ to interaction strength

ϵ	$\mathcal{N}_{\max}/\epsilon$
0.1	0.99197
0.05	0.99782
0.01	0.99969
0.005	0.99974
0.001	0.99976
0.0005	0.99976
0.0001	0.99976

6.2.2 Strong Interaction

FIG. 6.3 shows evolution of entanglement in the partition $A - BC$ where the Hamiltonian is taken to be $H = aH_{\text{free}} + H_{\text{int}}$. H_{int} is interaction Hamiltonian given by the last term in Eq. (6.5) while H_{free} is free Hamiltonian given by the first three terms in Eq. (6.5). The coefficient of the free Hamiltonian is taken to be very small $a = 0.00001$ to allow strong interaction limit. In this limit, smaller ϵ leads to a delay in achieving maximum entanglement. The same value of entanglement amplitude can be achieved with smaller ϵ at the expense of time. In fact $\epsilon\, x_{\text{max}} \approx 1.8138$. The reason is as follows: evolution operator is given by $U = \exp(-iHx)$. The free Hamiltonian is negligible compared to the interaction Hamiltonian. So the interaction strength factor effectively serves as a coefficient for the whole Hamiltonian. This interaction strength factor can then be thought of as a scaling factor for x in the evolution operator resulting in a scaling in the evolution of entanglement as illustrated in FIG. 6.3. The overall behaviour in this limit is in contrast with the one in weak interaction limit.

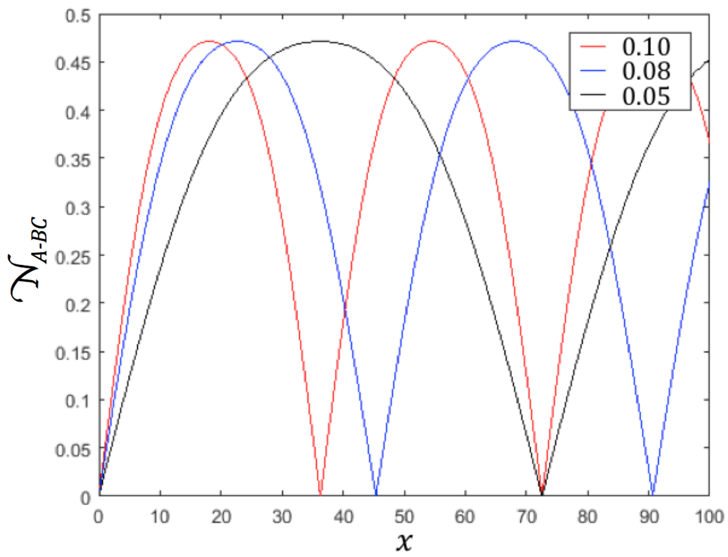

FIGURE 6.3: Negativity in the partition $A - BC$ as a function of dimensionless time for different values of interaction strength $\epsilon = 0.10, 0.08, 0.05$.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

In this thesis we have considered two methods of distributing entanglement with the help of an ancillary system. First we studied entanglement distribution where interactions are discrete and the state of the whole system is pure. We found some entanglement measures that are subadditive, i.e. entropy of entanglement and linear entropy, satisfy entanglement inequality that will lead to non-excessive distribution where entanglement gain is less than entanglement communicated. Negativity and logarithmic negativity also obey such inequality for some dimension of A. Distribution is non-excessive if the dimension of A is less than 3 for negativity measure and less than 4 for logarithmic negativity measure. Violations of the entanglement inequality can always be found for higher dimension of A leading to excessive distribution in which entanglement gain is higher than entanglement communicated. This shows that excessive distribution is possible even in pure states. If entanglement distribution is non-excessive and the discrete interaction method is applied a number of times to maximise entanglement, entanglement grow is bounded by the sum of entanglement in the partition $C - AB$

Next we studied entanglement distribution where interactions are continuous. We have proven that entanglement between A and B cannot grow if the state of the whole system is pure and the ancilla is separable. This implies that if there is entanglement grow between A and B, C has to be entangled with AB. The same result applies to entanglement in the partition $A - BC$ if the state of the whole system is mixed but the state of C is constant and in a product form as in the Born approximation. This implies that correlation in the partition $C - AB$ is needed to distribute entanglement. We also computed entanglement as a function of time in some circumstances. We have shown that perturbation theory can create an artefact in which entanglement grow between A and B is possible in pure states with separable C. We investigated entanglement in weak and strong interaction limit. Our findings show: In weak interaction limit, maximum entanglement is achieved at the same time for smaller interaction strength at the expense of amplitude; the amplitude is proportional to the interaction strength. While in strong interaction limit, the same entanglement amplitude can be achieved for smaller interaction strength at the expense of time; the time it takes to achieve maximum entanglement is inversely proportional to the interaction strength. This shows that, depending on the nature of interaction, maximum entanglement and time needed can be controlled by varying the interaction strength.

7.2 Future Work

There are some unanswered questions and new possibilities in this thesis that can be explored further. Therefore this Section will propose some future work for further investigation.

7.2.1 Continuous Interaction Method: Non-classical C?

It has been proven that non-classical correlation in the partition $C-AB$ is needed to distribute entanglement between A and B for pure states. It has also been proven in this thesis that entanglement cannot grow for mixed states if C is in a product form, which means that there has to be correlation in the partition $C-AB$ to distribute entanglement. However the question remains whether the correlation needed has to be non-classical. In particular we are interested to obtain an example where there is entanglement grow while C is separable. Then we are to investigate whether quantum discord is the key to distribute entanglement. If so, there is a question whether it bounds entanglement grow.

7.2.2 Gravity as ancilla?

In this thesis, an example of Hamiltonian in which the ancilla is taken to be the vibrational mode (ion-trap experiment) or cavity mode (cavity-QED experiment) was used. Possibility of using gravity field as the ancilla can be explored. The first step would be to quantise gravity. Then find a system to which this quantised gravity field can be coupled. Finally, one can construct similar entanglement distribution schemes as the ones presented in this thesis.

Appendix A

Trotter Expansion

Theorem A.1. *For arbitrary $n \times n$ real or complex matrices A and B, Trotter expansion reads:*

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n \tag{A.1}$$

Proof. We will use Baker-Campbell-Hausdorff formula:

$$\log e^A e^B = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] + \cdots \tag{A.2}$$

and the limit identity given by $\lim_x e^{f(x)} = e^{\lim_x f(x)}$.

Taking the log of $(e^{A/n}e^{B/n})^n$ and using Eq. (A.2) will give

$$n \log e^{\frac{A}{n}} e^{\frac{B}{n}} = n\left(\frac{A}{n} + \frac{B}{n} + \frac{1}{2n^2}[A, B] + \frac{1}{12n^3}[A, [A, B]] + \frac{1}{12n^3}[B, [B, A]] + \cdots\right) \tag{A.3}$$

Take the exponent and limit $n \to \infty$ of Eq. (A.3):

$$\lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n = \lim_{n\to\infty} e^{\left(A+B+\frac{1}{2n}[A,B]+\frac{1}{12n^2}[A,[A,B]]+\frac{1}{12n^2}[B,[B,A]]+\cdots\right)} \tag{A.4}$$

The Trotter formula follows if we use the limit identity to the R.H.S of Eq. (A.4)

□

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