ON THE VIOLATION OF CHSH INEQUALITY

SUBMITTED BY
NEO YIMING

PHYSICS AND APPLIED PHYSICS
SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

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Abstract

The Bell's inequality is derived under assumption of the local hidden variable models. Therefore all non-entangled particles satisfy Bell's inequality, as I will briefly argue. However, bi-partite entangled states can violate Bell's inequality. It has been shown to be true for bi-partite maximally entangled states. I will rederive this so-called Gisin's theorem, in a slightly different way.

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To Stephen Hawking for his motivational quote “Remember to look up at the stars and not down at your feet. Try to make sense of what you see and wonder about what makes the universe exist. Be curious.” It keeps me going.

Contents

1 Introduction

In 1964 John Bell has shown that certain quantum predictions cannot be understood on the basis of general classical-like theories, so called local hidden variable models. He assumed local hidden variables are carried by particles, and came up with an inequality to describe this. The inequality is named after him, the Bell's inequality. Bell's inequality can be verified through experiments. Recently, in 2015, there were three loophole-free Bell test experiments within three months by independent groups [1][2][3]. We can safely say that in its roots nature is not in agreement with local hidden variable models.

Bell's inequality can also be expressed in the form of Clauser, Horne, Shimony, and Holt (CHSH) inequality. It has been shown that CHSH inequality is violated by all pure entangled states [4][5]. I will review this and conclude the proof in a slightly different way.

2 Review on some aspects of quantum formalism

2.1 Condition for entanglement

Quantum entanglement is a phenomenon where, even though two or more objects are separated from each other in space, the quantum states that describes them can only be attributed to all particles.

A bi-partite state $|\psi\rangle_{AB}$ is entangled if

$$|\psi\rangle_{AB} \neq |\psi\rangle_A |\psi\rangle_B . \tag{1}$$

So the following state

$$\begin{aligned} |\psi\rangle_{AB} &= CE|00\rangle + CF|01\rangle + DE|10\rangle + DF|11\rangle \\ &= (C|0\rangle + D|1\rangle) \otimes (E|0\rangle + F|1\rangle), \end{aligned} \tag{2}$$

is an example of non-entangled state as we explicitly factorized it into $|\psi\rangle_A$

and $|\psi\rangle_B$. A handy way of decoding whether a pure state is entangled is presented in the next section.

2.2 Schmidt decomposition

Schmidt decomposition is a form of expressing a vector in a tensor product space [6]. Let $|\psi\rangle$ be a pure state of a system AB, which can be given as:

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle |k\rangle \tag{3}$$

where $|j\rangle$ and $|k\rangle$ are any orthonormal bases for systems A and B respectively, and a is a matrix of complex numbers a_{jk}. Using singular value decomposition, $a = UDV^+$, where D is a diagonal matrix with non-negative elements, and U and V^+ are unitary matrices. Hence,

$$|\psi\rangle = \sum_{ijk} U_{ji} D_{ii} V_{ik}^+ |j\rangle |k\rangle \,. \tag{4}$$

Define $|i_A\rangle = \sum_j U_{ji} |j\rangle \,, |i_B\rangle = \sum_k V_{ik}^+ |k\rangle \,, \lambda_i = D_{ii}$, the Schmidt decomposition of $|\psi\rangle$ is:

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \,. \tag{5}$$

In other words, there exist orthonormal states $|i_A\rangle$ and $|i_B\rangle$ for systems A and B respectively and real coefficients λ_i (also called the Schmidt coefficient). If $|\psi\rangle$ has at least two non-zero Schmidt coefficients, then $|\psi\rangle$ is entangled.

For example, let us find out if the following state is entangled:

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle). \tag{6}$$

We can use the Schmidt decomposition. Through singular value decomposition,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad V^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Using $|i_A\rangle = \sum_{j=0}^{1} U_{ji} |j\rangle$, we get:

$$|0_A\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle , \quad |1_A\rangle = -\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle . \tag{8}$$

Using $|i_B\rangle = \sum_{k=0}^{1} V_{ik}^{+} |k\rangle$, we get:

$$|0_B\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle , \quad |1_B\rangle = -\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle . \tag{9}$$

Using $\lambda_i = D_{ii}$, we get:

$$\lambda_0 = 1, \quad \lambda_1 = 0. \tag{10}$$

Substitute (8), (9), and (10) into (5),

$$|\psi\rangle = 1(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle)(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle). \tag{11}$$

Since $|\psi\rangle$ has only one non-zero Schmidt coefficient, it is clearly not an entangled state.

2.3 Density operator

The density operator describes arbitrary quantum states. In a mixed state, the system is in two or more states with their corresponding probabilities. Density operator ρ of mixed state is:

$$\rho = \sum_j P_j |\psi_j\rangle \langle\psi_j| , \tag{12}$$

where P_j is the probability.

In a pure state, the system is in only one state, with a corresponding probability of 1. Hence, the density operator of pure state is just a projection operator.

For a qubit, the density operator can also be expressed in the form:

$$\rho = \frac{1}{2}(\sigma_0 + \vec{s}.\vec{\sigma}), \tag{13}$$

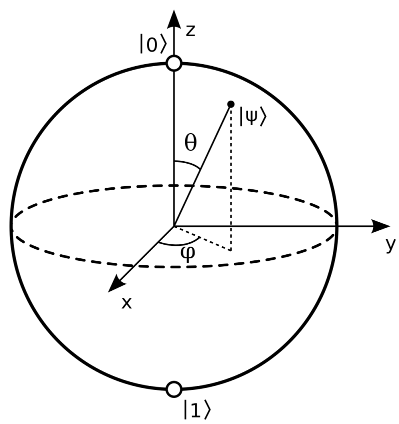

Figure 1: Bloch sphere. States on the surface are pure and inside are mixed.

where σ_0 is the identity matrix, $\vec{s}$ is vector with components (s_x, s_y, s_z) and $\vec{\sigma}$ is vector with components $(\sigma_x, \sigma_y, \sigma_z)$. $\vec{s}$ is called the Bloch vector. $\vec{\sigma}$ consists of Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

2.4 Bloch sphere

The Bloch sphere is a geometric representation of qubit states as points within a unit sphere (Fig. 1).

2.4.1 Pure state

A qubit is in a pure state if the corresponding Bloch vector lies on the surface of the Bloch sphere. The condition for purity is $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$ [7], where Tr is the trace of a matrix.

To prove that, we can evaluate $\mathrm{Tr}(\rho^2)$.

$$\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + \vec{r}^2). \tag{15}$$

While evaluating, we can use the expression $\mathrm{Tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}$ to assist in our calculation, where $\delta_{\mu\nu}$ is the Kronecker delta. When $\text{Tr}(\rho^2) = 1$, we get:

$$|\vec{r}| = 1. \tag{16}$$

Since $|\vec{r}| = 1$, the points all lie on the surface of the Bloch sphere, which means the qubit is in a pure state.

2.4.2 Mixed state

A qubit is in a mixed state if the Bloch vector lies within the Bloch sphere.

We can show that by evaluating the eigenvalues of the density operator. From the eigenvalues, we can determine the radius of the Bloch vectors.

$$\det(\rho - \lambda\sigma_0) = 0 \tag{17}$$

$$\lambda^2 - \lambda - \frac{\vec{r}^2}{4} + \frac{1}{4} = 0. \tag{18}$$

Solving the quadratic equation, we get:

$$\lambda_1 = \frac{1 + |\vec{r}|}{2}, \quad \lambda_2 = \frac{1 - |\vec{r}|}{2}. \tag{19}$$

The eigenvalues are non negative, therefore:

$$0 \leq |\vec{r}| < 1. \tag{20}$$

For a mixed state, the radius is between 0 to 1, which means the Bloch vector lies within the Bloch sphere.

2.4.3 Opposite vectors

Let $\rho_1 = \frac{1}{2}(\sigma_0 + \vec{r}.\vec{\sigma})$ and $\rho_2 = \frac{1}{2}(\sigma_0 + \vec{s}.\vec{\sigma})$. When $\text{Tr}(\rho_1\rho_2) = 0$, density operators ρ_1 and ρ_2 are orthogonal to each other while the Bloch vectors $\vec{r}$ and $\vec{s}$ are in opposite direction on the Bloch sphere.

We can show this by evaluating the following:

$$\text{Tr}(\rho_1\rho_2) = \frac{1}{2}(1 + \vec{r}.\vec{s}). \tag{21}$$

Again, we use $\mathrm{Tr}(\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu}$ to assist in our calculation. When $\mathrm{Tr}(\rho_1\rho_2) = 0$, we get:

$$|\vec{r}|.|\vec{s}| = -1. \tag{22}$$

The dot product can be expressed as another form, $|\vec{r}||\vec{s}|\cos\theta = -1$. Since $|\vec{r}|$ and $|\vec{s}|$ are equal to 1, we get:

$$\theta = \pi, \tag{23}$$

where θ is the angle between the two vectors. We see that $\vec{r}$ and $\vec{s}$ points in opposite direction. In other words, $\vec{r} = -\vec{s}$.

2.5 Measurement

Consider an observable having two measurement results with (eigenvalues) equal to ± 1. The operator for the measurement is:

$$\begin{aligned} M &= (+1)\left|m_+\right\rangle\left\langle m_+\right| + (-1)\left|m_-\right\rangle\left\langle m_-\right| \\ &= \frac{1}{2}(\sigma_0 + \vec{m}_+.\vec{\sigma}) - \frac{1}{2}(\sigma_0 + \vec{m}_-.\vec{\sigma}). \end{aligned} \tag{24}$$

Since $\vec{m}_-$ is opposite to $\vec{m}_+$, $\vec{m}_- = -\vec{m}_+$. Hence,

$$M = \vec{m}_+.\vec{\sigma}, \tag{25}$$

where $|\vec{m}_+| = 1$.

3 Violation of CHSH inequality by pure entangled states

3.1 CHSH inequality

Consider two observers Alice and Bob, and a source that produces two-particle states. One particle is sent to Alice, who can choose between two observables a and a' (a and a' can stand for angular momentum measurements along two different axes). The other particle is sent to Bob, who can

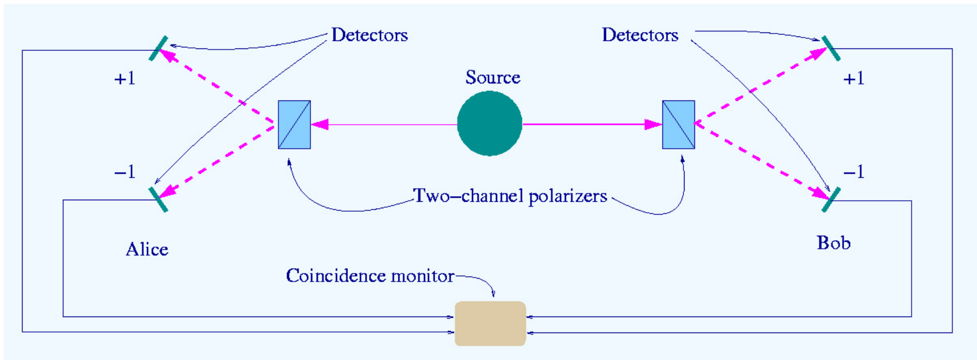

Figure 2: Bell test experiment.

choose between two observables b and b'. We will let the measurement results be ± 1, in other words $a, a', b, b' = \pm 1$. This is the Bell test experiment (Fig. 2).

In order to derive the Bell-CHSH inequality, we start with

$$(a + a')b + (a - a')b' = \pm 2. \tag{26}$$

If $(a + a') = 0$, $(a - a') = \pm 2$ and if $(a - a') = 0$, $(a + a') = \pm 2$. We obtain the inequality by averaging (26) over many experimental runs [8].

$$S = E(a, b) - E(a, b') + E(a', b) + E(a', b') \leq 2, \tag{27}$$

where, for example $E(a, b)$ is the average of the product of local results $a.b$.

3.2 Local hidden variable model

In local hidden variable model, the particles obey locality (the measurement results at Alice are independent of any actions in Bob's lab) and admit hidden variables [9]. Hidden variables can be anything not present in quantum formalism. Mathematically, they can take the form of numbers, vectors, matrices, etc. Since Bell's inequality is based on the local hidden variable model, it is satisfied by all local hidden variable models.

Within local hidden variable model, the probability that Alice observes a when she measures observable x and Bob observes b when he measures observable y is given by:

$$P(a,b|x,y) = \sum_{\lambda} P_{\lambda} P(a|x,\lambda) P(b|y,\lambda), \tag{28}$$

where λ denotes hidden variables and P_{λ} is their distribution.

3.3 All non-entangled particles satisfy Bell's inequality

If particles are not entangled, their state can be expressed as

$$\rho = \sum_{j} P_j \alpha_j \otimes \beta_j, \tag{29}$$

where P_j are probabilities and α_j, β_j are density matrices.

We will evaluate $P_{QM}(a,b|x,y)$ measured on ρ:

$$P_{QM}(a,b|x,y) = \mathrm{Tr}(\rho \left|a_x\right\rangle \left\langle a_x\right| \otimes \left|b_y\right\rangle \left\langle b_y\right|). \tag{30}$$

Substitute (29) into (30),

$$\begin{aligned} P_{QM}(a,b|x,y) &= \sum_{j} P_j \mathrm{Tr}((\alpha_j \otimes \beta_j)(\left|a_x\right\rangle \left\langle a_x\right| \otimes \left|b_y\right\rangle \left\langle b_y\right|)) \\ &= \sum_{j} P_j \mathrm{Tr}(\alpha_j \left|a_x\right\rangle \left\langle a_x\right| \otimes \beta_j \left|b_y\right\rangle \left\langle b_y\right|) \\ &= \sum_{j} P_j \mathrm{Tr}(\alpha_j \left|a_x\right\rangle \left\langle a_x\right|).\mathrm{Tr}(\beta_j \left|b_y\right\rangle \left\langle b_y\right|) \\ &= \sum_{j} P_j P(a|x,\alpha_j) P(b|y,\beta_j). \end{aligned} \tag{31}$$

By choosing $\lambda = (j, \alpha_j, \beta_j)$ and $p_{\lambda} = p_j$, Eq. (31) is shown to have the form

$$P_{QM}(a,b|x,y) = \sum_{\lambda} P_{\lambda} P(a|x,\lambda) P(b|y,\lambda), \tag{32}$$

familiar from local hidden variable model. Therefore, all non-entangled particles satisfy Bell's inequality.

3.4 Expectation value

In order to evaluate quantum value of Bell's parameters for entangled states, we will need the following considerations. In quantum mechanics, the measurement made by Alice, M_A, and Bob, M_B, can be expressed as

$$M_A = \vec{a_K}.\vec{\sigma}, \quad M_B = \vec{b_L}.\vec{\sigma}, \tag{33}$$

where K and L can be 1 or 2, refer (25).
The eigenstates of these observables are:

$$\begin{aligned} |+\rangle_A \langle +| = \frac{1}{2}(\sigma_0 + \vec{a}_K.\vec{\sigma}), \quad |-\rangle_A \langle -| = \frac{1}{2}(\sigma_0 - \vec{a}_K.\vec{\sigma}), \\ |+\rangle_B \langle +| = \frac{1}{2}(\sigma_0 + \vec{b}_L.\vec{\sigma}), \quad |-\rangle_B \langle -| = \frac{1}{2}(\sigma_0 - \vec{b}_L.\vec{\sigma}). \end{aligned} \tag{34}$$

The general form of expectation value is given by:

$$\begin{aligned} E(\vec{a_K}, \vec{b_L}) =& P(++) + P(--) - P(+-) - P(-+) \\ &= \langle\psi| ++\rangle \langle ++ |\psi\rangle + \langle\psi| --\rangle \langle -- |\psi\rangle - \\ &\quad \langle\psi| +-\rangle \langle +- |\psi\rangle - \langle\psi| -+\rangle \langle -+ |\psi\rangle \\ &= \langle\psi| M_A \otimes M_B |\psi\rangle. \end{aligned} \tag{35}$$

3.5 All pure entangled states of two qubits violate CHSH inequality

The general state of two qubits can be written in the Schmidt decomposition

$$|\psi\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle, \quad \alpha \in \left[0, \frac{\pi}{4}\right], \tag{36}$$

where we define $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Let us consider measurements in the XZ plane:

$$M_A = \sin(a_K).\sigma_x + \cos(a_k).\sigma_z, \quad M_B = \sin(b_L).\sigma_x + \cos(b_L).\sigma_z. \tag{37}$$

Substitute (37) into (35), the expectation value is:

$$\begin{aligned} E(\vec{a_K},\vec{b_L}) =& \sin(a_K)\sin(b_L)\langle\psi|\sigma_x\otimes\sigma_x|\psi\rangle + \sin(a_K)\cos(b_L)\langle\psi|\sigma_x\otimes\sigma_z|\psi\rangle + \\ & \cos(a_K)\sin(b_L)\langle\psi|\sigma_z\otimes\sigma_x|\psi\rangle + \cos(a_K)\cos(b_L)\langle\psi|\sigma_x\otimes\sigma_x|\psi\rangle. \end{aligned} \tag{38}$$

From the Schmidt decomposition,

$$\begin{aligned} \langle\psi|\sigma_x\otimes\sigma_x|\psi\rangle = \sin(2\alpha), \quad \langle\psi|\sigma_x\otimes\sigma_z|\psi\rangle = 0, \\ \langle\psi|\sigma_z\otimes\sigma_x|\psi\rangle = 0, \quad \langle\psi|\sigma_x\otimes\sigma_x|\psi\rangle = 1. \end{aligned} \tag{39}$$

Substitute (39) into (38), the expectation value is now:

$$E(\vec{a_K},\vec{b_L}) = \cos(a_K)\cos(b_L) + \sin(2\alpha)\sin(a_K)\sin(b_L). \tag{40}$$

Substitute (40) into the left-hand side of the CHSH inequality in Eq. (27),

$$\begin{aligned} S =& \cos(a_K)[\cos(b_L)+\cos(b'_L)] + \cos(a'_K)[\cos(b_L)-\cos(b'_L)] + \\ & \sin(2\alpha)\{\sin(a_K)[\sin(b_L)+\sin(b'_L)] + \sin(a'_K)[\sin(b_L)-\sin(b'_L)]\}. \end{aligned} \tag{41}$$

We note that this can be expressed in the form of four vectors. The vectors are

$$\begin{aligned} \vec{v_1} &= (\cos(a_K), \sin(a_K)), \\ \vec{v_2} &= (\cos(a'_K), \sin(a'_K)), \\ \vec{w_1} &= (\cos(b_L)+\cos(b'_L), \sin(2\alpha)[\sin(b_L)+\sin(b'_L)]), \\ \vec{w_2} &= (\cos(b_L)-\cos(b'_L), \sin(2\alpha)[\sin(b_L)-\sin(b'_L)]). \end{aligned} \tag{42}$$

Hence,

$$S = \vec{v_1}.\vec{w_1} + \vec{v_2}.\vec{w_2}. \tag{43}$$

The sum of dot products of vectors is less than or equal to the sum of magnitude of vectors:

$$S = |\vec{v_1}||\vec{w_1}|\cos(\alpha) + |\vec{v_2}||\vec{w_2}|\cos(\alpha) \leq |\vec{v_1}||\vec{w_1}| + |\vec{v_2}||\vec{w_2}|, \tag{44}$$

and this bound can be achieved as $\vec{v_1}$ and $\vec{v_2}$ are arbitrary unit vectors, so they can always be put parallel to $\vec{w_1}$ and $\vec{w_2}$. We have:

$$\begin{aligned} S = |\vec{v_1}||\vec{w_1}| + |\vec{v_2}||\vec{w_2}| =& \sqrt{[\cos(b_L)+\cos(b'_L)]^2 + \sin^2(2\alpha)[\sin(b_L)+\sin(b'_L)]^2} + \\ & \sqrt{[\cos(b_L)-\cos(b'_L)]^2 + \sin^2(2\alpha)[\sin(b_L)-\sin(b'_L)]^2}. \end{aligned} \tag{45}$$

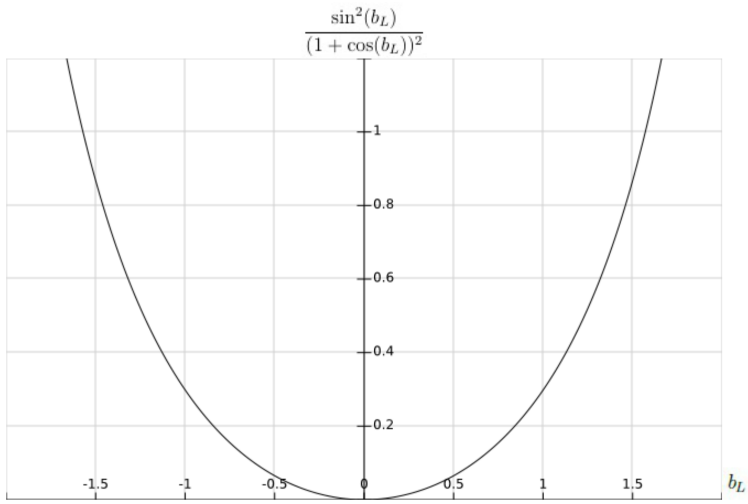

Figure 3: Graph of right-hand side of (48) against b_L.

We can choose a starting point of measurement from anywhere, so we let $b' = 0$ without loss of generality.

$$S = |\vec{v_1}||\vec{w_1}| + |\vec{v_2}||\vec{w_2}| = \sqrt{[1+\cos(b'_L)]^2 + \sin^2(2\alpha)\sin^2(b_L)} + \sqrt{[1-\cos(b'_L)]^2 + \sin^2(2\alpha)\sin^2(b_L)}. \tag{46}$$

This can be expressed in the form of the following two vectors:

$$\vec{x} = (1, \sin^2(2\alpha)), \quad \vec{y} = ((1+\cos(b_L))^2, \sin^2(b_L)). \tag{47}$$

For the dot product of these two vectors to have the greatest value, they must be aligned parallel to each other, in other words

$$\sin^2(2\alpha) = \frac{\sin^2(b_L)}{(1+\cos(b_L))^2}. \tag{48}$$

From Fig. 3, the graph is continuous in the range from 0 to 1 on the y-axis. This means that there is always a b_L that solves the equation. Substitute

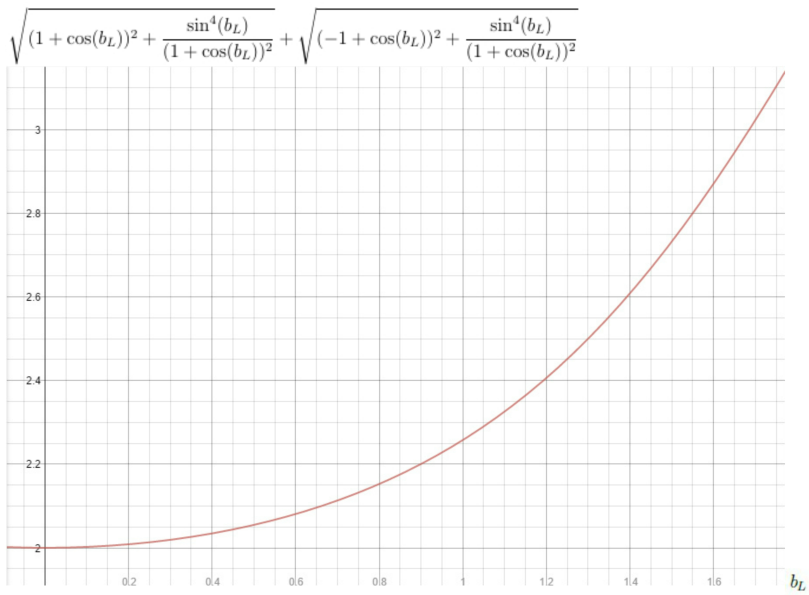

Figure 4: Graph of right-hand side of (49) against b_L.

(48) into (46), we get:

$$S = |\vec{v_1}||\vec{w_1}| + |\vec{v_2}||\vec{w_2}| = \sqrt{(1+\cos(b_L))^2 + \frac{\sin^4(b_L)}{(1+\cos(b_L))^2}} + \sqrt{(-1+\cos(b_L))^2 + \frac{\sin^4(b_L)}{(1+\cos(b_L))^2}}. \tag{49}$$

We need to check b_L from the range 0 to $\frac{\pi}{2}$ since the range of α is from 0 to $\frac{\pi}{4}$. From Fig. 4, we see that (49) is always greater than 2 in this range. Therefore, CHSH inequality is violated for all bi-qubit entangled pure states.

3.6 Restricted local hidden variable model

We have shown that all entangled pure states violate a Bell inequality and that all non-entangled states satisfy Bell inequalities. Here, we will like to ask if restricting local hidden variable models allows even non-entangled states to violate analog of Bell inequality.

Case	a_+	b_+	b_-
1	$\pm r_A$	± 1	$\pm r_B$
2	$\pm r_A$	$\pm r_B$	± 1

Table 1: Measurement results

As the simplest case, we consider a source which sends one bit to Alice and one bit to Bob. Let the bit sent to Alice be $r_A = \pm 1$ and that to Bob be $r_B = \pm 1$. Alice can choose between two settings, $x = \pm 1$ and Bob can choose between two settings, $y = \pm 1$. The measurement results of Alice can be calculated using one of four functions $a_x(r_A)$ and similarly for Bob $b_y(r_B)$:

$$a_x(r_A) = \pm 1, \pm r_A, \quad b_y(r_B) = \pm 1, \pm r_B. \tag{50}$$

The expectation value is:

$$\begin{aligned} E_{xy} &= \sum_{a_x, b_y} a_x b_y P(a_x, b_y) \\ &= \sum_{r_A, r_B, a, b} = a_x(r_A) b_y(r_B) P(a_x(r_A) = a, b_y(r_B) = b | r_A, r_B).P(r_A, r_B) \\ &= \overline{a_x b_y}. \end{aligned} \tag{51}$$

Thus,

$$E_{xy} = \overline{(\pm 1 \,\text{or} \pm r_A).(\pm 1 \,\text{or} \pm r_B)}. \tag{52}$$

Consider only the correlation functions that satisfy:

$$E_{++} \neq \pm 1, \quad E_{+-} \neq \pm 1, \quad E_{++} \neq E_{+-} \neq -E_{+-}. \tag{53}$$

The only non-trivial choices of functions $a_x(r_A)$ and $b_y(r_B)$ are gathered in Table 1.

In case 1, the result of b_+ is fixed at ± 1. We obtain:

$$E_{++} = \pm \overline{r_A}, \quad E_{+-} = \pm \overline{r_A r_B}. \tag{54}$$

In case 2, the result of b_- is fixed at ± 1. We obtain:

$$E_{++} = \pm \overline{r_A r_B}, \quad E_{+-} = \pm \overline{r_A}. \tag{55}$$

The probability in terms of function of results of Alice and Bob is:

$$P(r_A, r_B) = \frac{1}{4}(1 + r_A E_{11} + r_B \overline{r_B} + r_A r_B E_{12}) \geq 0. \tag{56}$$

By setting $r_A = +1$ and $r_B = +1$, $r_A = +1$ and $r_B = -1$, $r_A = -1$ and $r_B = +1$, $r_A = -1$ and $r_B = -1$, we get the four equations respectively.

$$\begin{aligned} 1 + E_{++} + \overline{r_B} + E_{+-} \geq 0, \\ 1 + E_{++} - \overline{r_B} - E_{+-} \geq 0, \\ 1 - E_{++} + \overline{r_B} - E_{+-} \geq 0, \\ 1 - E_{++} - \overline{r_B} + E_{+-} \geq 0. \end{aligned} \tag{57}$$

This will mean that the two correlation functions E_{++} and E_{+-} cannot be obtained by measuring particles which carry only one bit of hidden variable each.

4 Conclusion

In summary, I have reviewed the argument that the results of all possible measurements on non-entangled states will always satisfy Bell inequalities. I have also proven that all pure entangled states of two qubits violate the CHSH inequality, where this statement is known as Gisin's theorem.

An interesting future research direction is to consider hidden variable models with somewhat restricted hidden variables. For example, we could allow only finite number of bits to be carried as hidden variables. In such cases, it is plausible that even non-entangled quantum states will violate an analog of Bell inequalities. Last section describes first steps in this direction.

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