FEATURES OF GENUNINE MULTIPARTITE QUANTUM ENTANGLEMENT

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SUBMITTED
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1 Abstract

In this report, we investigate the various properties of genuine multipartite entanglement (GME). The focus is on two investigations. Firstly, we are interested in vanishing correlations in genuine multipartite entangled systems. One might intuitively expect (extrapolating from a bipartite case) that GME is always accompanied by non-zero n-fold correlation functions. It has recently been shown that this is not the case and this report provides new examples of this phenomenon. Next we focus on the maximum dimension of the subspace containing solely GME states. This is interesting as states from such subspaces are robust to small perturbations, which is essential for example to measurement based quantum computation.

2 Introduction to geometry in quantum physics

2.1 Hilbert Space and Bloch Sphere

In the early part of the 20th century, the Hilbert space is formulated along with the birth of quantum mechanics, by the collaboration of various physicist and mathematicians. The creation of Hilbert spaces allows physicists to express physical states in quantum mechanics as sets of normalised vectors in Hilbert spaces, and measurements or observables as self-adjoint operators acting on the state. In general, Hilbert space is defined as a vector space with inner product defined as below:

$$|a|^2 = \langle a|a\rangle \tag{1}$$

where a is a vector.
In quantum mechanics, Hilbert space has several special properties such as:

- The inner product of 2 vectors denoted by $\langle\psi_1|\psi_2\rangle$ has to obey the following Cauchy-Schwartz inequality relation,

$$|\langle\psi_1|\psi_2\rangle| \leq ||\psi_1||||\psi_2|| \tag{2}$$

- The orthogonal projector in the Hilbert space is defined as $P_i = |\psi_i\rangle\langle\psi_i|$, where $\langle\psi_i|\psi_j\rangle = 0$ for $i \neq j$. Any normalized $|\Psi\rangle = \sum_i \langle\psi_i|\Psi\rangle |\psi_i\rangle$, such that,

$$\sum_i P_i = \sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1} \tag{3}$$

It is imperative to introduce a 'spinoff' of Hilbert space, the projective Hilbert space for later usage. The projective Hilbert space is derived from the projectivization of the Hilbert space, associating the Hilbert space to a projective space, where the elements in the projective space are one-dimesional subspaces of the Hilbert space. We begin first by defining a projective space. The projective space is the set of one-dimensional subspaces of a given vector space. In other words, the projective space P^n is the set of lines that pass through the origin of the space defined. As we are mainly interested in the application to quantum space, the focus will be on the complex projective space $\mathbb{C}P^n$, which is defined as the set of rays in the complex space of $\mathbb{C}^{n+1}$. Geometrically, $\mathbb{C}P^0$ is a point in the space while, $\mathbb{C}P^1$ is a 2-sphere. To illustrate this, consider a particle with spin $\frac{1}{2}$. Every pure state of the particle will correspond to a unique direction. This is isomorphic to a sphere, or $\mathbb{C}P^1$. This sphere is known as the Bloch sphere and the boundary of this sphere corresponds to the set of pure states. In this representation, the north $(\hat{z})$ and south $(\hat{-z})$ poles of the sphere are denoted by $|0\rangle$ and $|1\rangle$ repectively. $|0\rangle$ and $|1\rangle$ are representation of basis states of a 2 level quantum system describled by two-dimensional complex vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \& \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

2.2 Pure and Mixed states

In quantum mechanics, there are mainly 2 types of classification of quantum systems; Pure and Mixed. Any pure state of a single particle can be represented by any point in the Bloch sphere and for any n-particle composite system with non-entangled pure state $|\psi\rangle$, it can be decomposed into the following form,

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \ ... \ \otimes |\phi_n\rangle \tag{5}$$

This allow us to have complete information for each individual system and thus the exact state that each individual systems is in. Pure states also carry certain key properties:

- The normalised inner product of the state by its conjugate has to be 1.

$$\langle\psi|\psi\rangle = 1 \tag{6}$$

- The expectation value of a meaurement A on the pure state is:

$$\langle\psi|A|\psi\rangle = \langle A\rangle_\psi \tag{7}$$

- The expectation value can be expressed in terms of trace operator;

$$\langle A\rangle_\psi = \mathrm{Tr}(AP_\psi) \tag{8}$$

- The Von Neumann entropy of pure states is 0.

For illustration, suppose we are to describle the state $|\psi\rangle$, of the particle with spin $\frac{1}{2}$ such that there is 0.5 chance to find it with $\uparrow_z$ spin and 0.5 chance to find it with $\downarrow_z$ spin then;

$$|\psi\rangle = \frac{1}{\sqrt{2}}(a\,|\uparrow\rangle_z + b\,|\downarrow\rangle_z) \tag{9}$$

where a and b are coefficients and $|a|^2 = |b|^2$.

There are multiple solutions for a and b given the consideration for the phase of the states and an additional measurement will be required to find a unique set of solution for $|\psi\rangle$.For example, one can consider the solution where $a = 1$ and $b = i$ and compare it with the solution of $a = 1$ and $b = 1$. However, it can be noted that the set of solutions for a forms a 'ray' or one dimensional subspace in the Hilbert space, which can be represented as qelement in the projective space. The same treatment can also be applied to the set of b. The combination of the elements of a and b will form a *Variety* which will be discussed in the later portion. Physically, $|\psi\rangle$ is a superposition of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ and thus,

$$|\psi\rangle = |\uparrow\rangle_x \tag{10}$$

for $a = b = 1$.
Mixed states are generated from the mixing of pure states such that measurements cannot distingush which way the states were prepared. The mixed state is defined as:

$$\varrho = \sum_i^k p_i \, |\psi_i\rangle\langle\psi_i| \tag{11}$$

fulfilling the following conditions:

- $\varrho = \varrho^\dagger$
- $Tr(\varrho) = 1$
- $\varrho \geq 0$ or $\varrho^2 \geq \varrho$
- The Von Neumann entropy of mixed states is strictly positive.

Bennett [1] provided a simple case where mixed states can be generated. Suppose we have a state similar to (9) and noise is introduced into the system such that the system now has additional degrees of freedom. This will cause a non-unitary evolution of system describle by equation (9) into the mixed state ϱ_{AB}.

$$\varrho_{AB} = \frac{1}{\sqrt{2}}(a_2 \, |\uparrow_z\rangle\langle\uparrow_z| + b_2 \, |\downarrow_z\rangle\langle\downarrow_z|) \tag{12}$$

where a_2 and b_2 are coefficients.
Another difference comes when one takes a partial trace over B, on the density matrix ϱ_{AB}, one obtains the following relation:

$$\hat{\varrho}_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \frac{1}{2}\mathbb{1} \tag{13}$$

This is vastly different if applied to maximally entangled pure state from the case of a pure disentangled state.

2.3 Measurements in quantum systems

The expectation value of physical observables can be expressed in terms of density matrices as in (8) for mixed states. One can generalise (8), such it includes any generic mixed state. From (8), for an essemble of states,

$$\langle A \rangle = \sum_i p_i \, \mathrm{Tr}[A \, |\psi_i\rangle\langle\psi_i|] \tag{14}$$

or

$$\langle A \rangle = \mathrm{Tr}\left[A \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \right] \tag{15}$$

since the Tr operator is linear.
Another key tool of quantum measurement theory is the cyclic permutation of the Tr operator.

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB) \tag{16}$$

Suppose we do a measurement A on the system with state $|\psi_a\rangle$,

$$\langle A \rangle = \langle \psi_A | A | \psi_A \rangle = \mathrm{Tr}[|\psi_A\rangle\langle\psi_A| \, A] = \mathrm{Tr}[a \, |\psi_A\rangle\langle\psi_A|] \tag{17}$$

This allows us to perform and simplify certain operation in later discussions.

3 Entanglement

The field on Entanglement started in 1935 with the paper by Einstein, Podolsky and Rosen. The paper points out the incompleteness of quantum mechanics and the wave functions. The problem is restated and put in a different manner by Bohm in 1957. Consider a system with 0 net angular momentum that decays into 2 separate particles with spin $\frac{1}{2}$ each. The system is descible by the following equation:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_z - |\downarrow\uparrow\rangle_z) \tag{18}$$

Suppose after some time, the particles are far apart and a measurement is made on one of the particles, we assume has 2 possible measurement values, $\frac{\hbar}{2}\hat{z}$ or $-\frac{\hbar}{2}\hat{z}$. This means that the other particle must have values $-\frac{\hbar}{2}\hat{z}$ or $\frac{\hbar}{2}\hat{z}$ correspondingly. Thus the $\hat{z}$ component of the spin of both particle is strongly correlated. However, suppose one tried to measure the $\hat{x}$ spin of one of the particle, according to quantum mechanics the $\hat{z}$ component becomes less definite, due to the effects of the measurement on the system. Since the $\hat{z}$ spin of both particles are strongly correlated, the $\hat{z}$ spin of the second particle is also affected. The strange behaviour comes when we have already assumed that the particles are far apart. If the spin is not affected by the second measurement, then the second particle will have definite values for $\hat{x}$ and $\hat{z}$ component of the spin, which is inconsistent with Heisenberg's Uncertainty principle. To solve this problem, the idea of entanglement is introduced where the spins of the particlea are entangled such that the second measurement also affect the $\hat{z}$ component of the second particle, preventing it from having a definite value.

When particles are entangled, the quantum state of each individual particle can no longer be defined independently and requires the whole system to be described as a single state. For instance, for 2 entangled qubits in the Bell state, we have,

$$|\psi_{bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{19}$$

In this quantum state, there is equal probability of measuring $|00\rangle$ and $|11\rangle$. In a less convenient form, one can express it as,

$$|\psi_{bell}\rangle\langle\psi_{bell}| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

3.1 LOCC and SLOCC

LOCC, local operation and classical communication refers to set of rules that form the basis of entanglement theory. Suppose we have a state $|\psi_{locc}\rangle$ that is shared among 2 parties, Alice and Bob, such that both parties can perform arbitary measurements and operations on the local system. Next, restrict the communication channel between Alice and Bob to be classical. Then, under the rules of LOCC, the possible types of transformation operation and creation of the states are restricted. States that can be created by only using LOCC, are separable. One can make use of the rules of LOCC to create a set of steps and preparation methods to make conversion between certain states. Suppose there is a non-zero probability to convert from $|\psi_\alpha\rangle$ to $|\psi_\beta\rangle$ under LOCC. Then the operation in that convertion is known as stochastic local operation and classical communication, SLOCC. States that can be transformed into one another under SLOCC are called SLOCC equivalent.

3.2 Schmidt Rank

In the study of quantum information theory, one is mainly interested in 2 main issues. The former being how to quantify entanglement and latter being how many different types of entangement are there and how does one classify them. To ease the process of entanglement classification and characterization, one introduces a certain form of the state known as Schmidt decomposition.

For brevity, one can first consider a Hibert space H_{AB} that is composed of 2 subspaces, H_A and H_B such that $H_{AB} = H_A \bigotimes H_B$. Without the loss of generality, one asummes that $dim(H_A) \geq dim(H_B)$. Then, there exist orthonormal states $|i_A\rangle$ and $|i_B\rangle$ such that,

$$|\Psi\rangle = \sum_{i=1}^{dim(H_B)} \lambda_i |i_A\rangle |i_B\rangle \tag{21}$$

This statement is derived from mathematical factorisation of a matrix known as Singular Value Decomposition:
Given a $m \times n$ matrix M, it can be factorized into the following form, VDU^T where,

- V : set of orthonormal eigenvectors of MM^T
- U : set of orthonormal eigenvectors of M^TM
- D : singlar values of M.

Following, let $|k\rangle$ and $|l\rangle$ be any abitary fixed orthonormal bases of Hilbert space A and B respectively, then the state $|\Psi_s\rangle$ can be expressed as,

$$|\Psi_s\rangle = \sum_{k,l} \lambda_{kl} |k\rangle |l\rangle \tag{22}$$

Setting λ_{kl} as the matrix M, one can decompose it such that,

$$|\Psi_s\rangle = \sum_{k,l} VDU^T |k\rangle |l\rangle \tag{23}$$

Since D is a diagonal matrix, with elements $\{a_1.....a_K\}$ and setting $\sum_m U_{mk} |i\rangle$ as $|i_A\rangle$ and $\sum_n V^*_{kn} |j\rangle$ as $|i_B\rangle$, one obtains equation (22).

The number of non-zero diagonals is refered to as the Schmidt rank of the system. In other words, Schmidt rank is the number of coefficient in the decomposition. If the Schmidt rank of state $|\Psi\rangle$ of a composite system is 1, then the state is a product state (separable state), which is expressed as:

$$|\Psi_s\rangle = |\psi\rangle \otimes |\phi\rangle \tag{24}$$

In other words, for a state to be entangled, its Schmidt rank has to be strictly more than 1. One can also generalise Schmidt rank defined above to include mixed states using convex hulls.

4 Genuine Multipartite Entanglement

A pure state is genuine multipartite entangled(GME) iff all partitions produce mixed density matrices. Mixed reduced density matrix refers to the partial trace of density matrix or the marginal. To put it in a more formal manner, a system is separable if it can be written as a convex combination of product states. Let's illustrate this with simple cases of bipartite and tripartite systems.

Consider a bipartite system ϱ_{AB}, the system is separable iff,

$$\varrho_{AB} = \sum_i p_i \varrho_A^i \otimes \varrho_B^i \tag{25}$$

For system with more parties, there can be many different partitions for separation. However in this context, we are only interested in bipartition i.e divisible into two parts. We shall then use a tripartite system as an example.

A tripartite system is fully separable iff,

$$\varrho_{ABC} = \sum_i p_i \varrho_A^i \otimes \varrho_B^i \otimes \varrho_C^i \tag{26}$$

And biseparable iff,

$$\varrho_{ABC} = p_1 \varrho_{A|BC} + p_1 \varrho_{B|AC} + p_1 \varrho_{C|AB} \tag{27}$$

with

$$\varrho_{A|BC} = \sum_i p_i \varrho_A^i \otimes \varrho_{BC}^i \tag{28}$$

and so forth.
For instance, a GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is genuine multipartite entangled since one cannot express it in the form (27). On the other hand, the state, $|\psi_{tri}\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |111\rangle)$ is biseparable, as $|\psi_{tri}\rangle$ can be express in the following form:

$$|\psi_{tri}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|11\rangle) \tag{29}$$

$$|\psi_{tri}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \otimes (|1\rangle) \tag{30}$$

For the case of mixed states, the verification of separability is not as simple. For a bipartite case, one has the Peres-Horodecki criterion (PPT criterion) [2] which is a necessary and sufficient condition for a system to be separable with dimensions 2×2 and 2×3. Suppose we have a bipartite system with a density matrix ϱ_{ppt}. Then one can express the matrix in terms of a chosen product basis;

$$\varrho_{ppt} = \sum_{i,j}^{N} \sum_{k,l}^{M} \varrho_{ij,kl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{31}$$

If one does a partial transpose on the state (on one subspace), then the transposed state is defined as:

$$\varrho^{T_1} = \sum_{i,j}^{N} \sum_{k,l}^{M} \varrho_{ji,kl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{32}$$

If one performs the same operation on the other subspace, then the other transposed state is defined as:

$$\varrho^{T_2} = \sum_{i,j}^{N} \sum_{k,l}^{M} \varrho_{ij,lk} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{33}$$

If ϱ^{T_1} or ϱ^{T_2} is positive then, the matrix is said to be PPT and any PPT matrix is separable in dimensions 2×2 and 2×3. For the multipartite case, there is no such simple process. One needs to have many different criteria to aid in the detection of GME. To better understand the problem, one can consider Dicke states which are always GME.

$$|D_n^m\rangle = \frac{1}{\sqrt{\binom{m}{n}}} \sum_{\{\alpha\}} \left|D_{\{\alpha\}}\right\rangle \tag{34}$$

where,

$$\left|D_{\{\alpha\}}\right\rangle = \bigotimes_{i \neq \{\alpha\}} |0\rangle_i \bigotimes_{i = \{\alpha\}} |1\rangle_i \tag{35}$$

For instance, for n = 4, m = 2, we have 6 permutations of 1s and 0s:

- $\left|D_{\{12\}}\right\rangle = |1100\rangle$
- $\left|D_{\{13\}}\right\rangle = |1010\rangle$
- $\left|D_{\{14\}}\right\rangle = |1001\rangle$
- $\left|D_{\{23\}}\right\rangle = |0110\rangle$
- $\left|D_{\{24\}}\right\rangle = |0101\rangle$
- $\left|D_{\{34\}}\right\rangle = |0011\rangle$

Having our testing states, we can start defining various testing criteria or 'Witness'.

4.1 Criterion

The whole criterion or witness relies on the strict requirement of convexity. In other words, when one mixes different separable states, another separable state will be obtained. Most witnesses are necessary but non sufficient. As such, a non violation of the witness does not equate to the state being a separable or biseparable. Moreover when noise is introduced to the system, different witnesses will have different ability to function properly. We can define a simple noise threshold by mixing a simple noise matrix $p \times \mathbb{1}$ in to a state that had been proven to be entangled. The threshold can then be determined by solving for p. Another issue with using entanglement is that generally, the minimum of 2 convex function is not instantly a convex function. This will lead to some complication when one transit from a bipartite system to that of a multipartite system. That being said, there are some criteria that can be generalised from the bipartite cases, such as:

- Range criterion:
  The range criterion states that a state ϱ is separable if there exist a set of product vectors that spans the range of ϱ. However, the noise treshold for this criterion is very low.

- Matrix realignment criterion:
  The realignment criterion is based on the realigned matrix that is constructed from the density matrix of a bipartite system. It is especially useful in distingushing bound entangled states from separable states. For instance Bennett et at [1] 3×3 inseparable states and Horodeki's 3×3 bound entangled states [2] were successfully tested by Chen [3].

  The theorem states that given a $A \times B$ bipartite density matrix ϱ_{AB} that is separable, then sum of all the singular values of the $A^2 \times B^2$ realigned density matrix is less or equal to 1. As a result, a bipartite pure state is separable iff the realigned matrix is equal to 1.

- Linear contraction criterion:
  This is an extension from the Matrix realignment criterion by Horedecki [4]. Suppose we have a density matrix similar to (32),

$$\varrho = \sum_{ij,kl} \varrho_{ijkl} \, |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{36}$$

  where ϱ_{ijkl} is a $d_1^2 \times d_2^2$ matrix. The trace norm of the matrix ϱ_{ijkl} for a pure state will be equal to 1 after normalization. Any permutation of the indices of ϱ_{ijkl} will also give the same results. Then, at least in one permutation, the dimension of the matrix will be $(d_1 \times d_2) \times (d_1 \times d_2)$. If the state is separable,

$$\mathrm{Tr}[||\varrho_{ijkl}||] = \sum_a p_a \, \mathrm{Tr}[||\Pi_p \varrho_{ijkl}||] \tag{37}$$

  where Π_p is the permuation of $ijkl$.
  The end results are similar to the PPT criterion,

$$\mathrm{Tr}[||\varrho_{ijkl}||] \leq 1 \tag{38}$$

$$\mathrm{Tr}[||\varrho_{ikjl}||] \leq 1 \tag{39}$$

  This is further expanded to include multipartite cases in [1511.00375v1].

- For 3 qubit systems, some important observations and witnesses were made by Guhne and Seevinck [5]. The first observation is that for a biseparable tri-partite system ϱ,

$$|\varrho_{1,8}| \leq \sqrt{\varrho_{2,2}\varrho_{7,7}} + \sqrt{\varrho_{3,3}\varrho_{6,6}} + \sqrt{\varrho_{4,4}\varrho_{5,5}} \tag{40}$$

To see how this is true, assume a bi-separable pure tripartite state $|\psi_{bi}\rangle = (a|0\rangle + b|1\rangle) \otimes (c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle)$. Then form the density matrix,

$$\varrho = \begin{pmatrix} a^2c_1^2 & a^2c_1c_2 & a^2c_1c_3 & a^2c_1c_4 & abc_1^2 & abc_1c_2 & abc_1c_3 & abc_1c_4 \\ a^2c_1c_2 & a^2c_2^2 & a^2c_2c_3 & a^2c_2c_4 & abc_1c_2 & abc_2^2 & abc_2c_3 & abc_2c_4 \\ a^2c_1c_3 & a^2c_2c_3 & a^2c_3^2 & a^2c_3c_4 & abc_1c_3 & abc_2c_3 & abc_3^2 & abc_3c_4 \\ a^2c_1c_4 & a^2c_2c_4 & a^2c_3c_4 & a^2c_4^2 & abc_1c_4 & abc_2c_4 & abc_3c_4 & abc_4^2 \\ abc_1^2 & abc_1c_2 & abc_1c_3 & abc_1c_4 & b^2c_1^2 & b^2c_1c_2 & b^2c_1c_3 & b^2c_1c_4 \\ abc_1c_2 & abc_2^2 & abc_2c_3 & abc_2c_4 & b^2c_1c_2 & b^2c_2^2 & b^2c_2c_3 & b^2c_2c_4 \\ abc_1c_3 & abc_2c_3 & abc_3^2 & abc_3c_4 & b^2c_1c_3 & b^2c_2c_3 & b^2c_3^2 & b^2c_3c_4 \\ abc_1c_4 & abc_2c_4 & abc_3c_4 & abc_4^2 & b^2c_1c_4 & b^2c_2c_4 & b^2c_3c_4 & b^2c_4^2 \end{pmatrix}$$

One can observe that $|\varrho_{1,8}| = abc_1c_4 = \sqrt{\varrho_{4,4}\varrho_{5,5}}$. This is only for the $A|BC$ partition. For $B|AC$ and $C|AB$ partition, the same logic applies. Since, the square root of 2 positive linear function is concave, the proof works for any mixture of states that have the bound above.
The next observation is that for any biseparable tripartite state,

$$|\varrho_{2,3}|+|\varrho_{2,5}|+|\varrho_{3,5}| \leq \sqrt{\varrho_{1,1}\varrho_{4,4}}+\sqrt{\varrho_{1,1}\varrho_{6,6}}+\sqrt{\varrho_{1,1}\varrho_{7,7}}+\frac{1}{2}(\varrho_{2,2}+\varrho_{3,3}+\varrho_{5,5}) \tag{41}$$

Using the density matrix ϱ above, one can observe that for partition $A|BC$, $|\varrho_{2,5}| = abc_1c_2 = \sqrt{\varrho_{1,1}\varrho_{6,6}}$ and that $|\varrho_{3,5}| = abc_1c_3 = \sqrt{\varrho_{1,1}\varrho_{7,7}}$. Lastly, $|\varrho_{2,3}| \leq \frac{1}{2}(\varrho_{2,2} + \varrho_{3,3})$. Combining the conditions developed for other partition, one obtain the criterion (42).

In our case we shall use another criterion that was developed by Guhne and Seevinck for the Dicke state. It is based on the first observation, and generalizing it to the multipartite cases.

$$\Xi^{|D_n^m\rangle}(\varrho) \leq \sqrt{\varrho_{0000}\varrho_{1111}} + \sum_{|I|=1}\sum_{|J|=3}\sqrt{\varrho_I\varrho_J} + \frac{3}{2}\sum_{|I|=2}\varrho_I \tag{42}$$

where $\Xi^{|D_n^m\rangle}(\varrho)$ is the sum of the absolute of the off-diagonal of ϱ in the left upper triangle and violation of this inequality means that the state is genuine multipartite entangled. To illustrate the inequality and the definition of the notations, consider the 4 partite Dicke state:

$$|D_2^4\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \tag{43}$$

Its density reads:

$$\varrho_{d4} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{44}$$

Hence,

$$\Xi^{|D_2^4\rangle}(\varrho) = |\varrho_{4,6}|+|\varrho_{4,7}|+|\varrho_{4,6}|+|\varrho_{4,7}|+|\varrho_{6,7}|+|\varrho_{7,6}|+|\varrho_{10,4}|+|\varrho_{11,4}|+|\varrho_{4,10}|+|\varrho_{4,11}| = \frac{5}{3} \tag{45}$$

$$\sqrt{\varrho_{0000}\varrho_{1111}} = \sqrt{\varrho_{1,1}\varrho_{16,16}} = 0 \tag{46}$$

We shall clarify the terms in equation (43) for future referencing. The term ϱ_{0000} refers to element $\varrho_{1,1}$ in the density matrix, while $\varrho_{1111} = \varrho_{16,16}$. ϱ_I is the set of indices that sums to $|I|$. For instance, in the set of $\varrho_{|1|}$ consist of $\varrho_{0001}, \varrho_{1000}, \varrho_{0100}, \varrho_{0010}$. Thus,

$$\sum_{|I|=1}\sum_{|J|=3}\sqrt{\varrho_I\varrho_J} = 0 \tag{47}$$

and

$$\sum_{|I|=2}\varrho_I = 1 \tag{48}$$

R.H.S of equation (43) giving a value of $\frac{3}{2}$, violating the inequality, proving that the states are genuine multiparite entangled.

The 4-partite W and GHZ states have slightly different criteria. Namely,

- For GHZ state $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, the criterion is given as:

$$\Xi^{GHZ_4}(\varrho) \leq \frac{1}{2}\sum_{|I|=1,2,3}\sqrt{\varrho_I\varrho_{\bar{I}}} \tag{49}$$

  The term $\varrho_{\bar{I}}$ refers to the terms with indices inverted with respect to ϱ_I. E.g. for $|\bar{1}|$ we have, $\varrho_{0111}, \varrho_{1011}\cdots$

- for W_4 state $|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, the criterion is given as:

$$\Xi^{W_4}(\varrho) \leq \sum_{|I|=2} \sqrt{\varrho_{0000}\varrho_I} + \sum_{|I|=1} \varrho_I \tag{50}$$

The main idea behind the criteria is the same as described above. It applied to all different partitions such as $A|BCD, AB|CD$. Note that states that violate any of the witness criteria will be a genuine multipartite entangled. We would have to derive the witness for the $\bar{W}$ for future reference.

4.2 Criterion for $\left|\bar{W_4}\right\rangle$

Let us begin by considering the following states, $|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$ and $\left|\bar{W_4}\right\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$ with density matrices denoted as ϱ_W and $\varrho_{\bar{W}}$ respectively.

$$\varrho_W = \begin{bmatrix}
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.
\end{bmatrix} \tag{51}$$

$$
\varrho_{\bar{W}} = \begin{bmatrix}
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.25 & 0. & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.
\end{bmatrix} \tag{52}
$$

The simplest approach will be mirroring the steps for $|W\rangle$. Let $\Xi_{\bar{W}}$ be the sum of the off diagonal lower triangular matrix elements of $\varrho_{\bar{W}}$ since they are mirror of one another. Then we can start to match term for term:

$$\varrho_{W_4}(1,1) = \varrho_{\bar{W}_4}(16,16) \tag{53}$$
$$\varrho_{W_4}(2,2) = \varrho_{\bar{W}_4}(15,15) \tag{54}$$
$$\varrho_{W_4}(3,3) = \varrho_{\bar{W}_4}(14,14) \tag{55}$$
$$\varrho_{W_4}(4,4) = \varrho_{\bar{W}_4}(13,13) \tag{56}$$
$$\varrho_{W_4}(5,5) = \varrho_{\bar{W}_4}(12,12) \tag{57}$$
$$\varrho_{W_4}(6,6) = \varrho_{\bar{W}_4}(11,11) \tag{58}$$
$$\varrho_{W_4}(7,7) = \varrho_{\bar{W}_4}(10,10) \tag{59}$$
$$\varrho_{W_4}(9,9) = \varrho_{\bar{W}_4}(7,7) \tag{60}$$
$$\varrho_{W_4}(10,10) = \varrho_{\bar{W}_4}(6,6) \tag{61}$$
$$\varrho_{W_4}(11,11) = \varrho_{\bar{W}_4}(5,5) \tag{62}$$
$$\varrho_{W_4}(12,12) = \varrho_{\bar{W}_4}(4,4) \tag{63}$$
$$\tag{64}$$

The corresponding inequality reads,

$$\Xi_{\bar{W}} \leq \sum_{|I|=2} \sqrt{\varrho_{1111}\varrho_I} + \sum_{|I|=3} \varrho_I \tag{65}$$

For comparison, one can conduct a noise test on both states, which should give the same value. Consider the following state, $\varrho_{W_4}(p) = (1-p)\,|W_4\rangle\langle W_4| + \frac{p}{16}\mathbb{1}$. The p value obtain by the Guhne is determined to be $p < \frac{4}{9}$. In our criteria, we obtain the same result for both $|W_4\rangle$ and $|\bar{W}_4\rangle$ which also correspond to that of Guhne. We shall use this set of criteria in the later stage.

5 Correlation

In Statistics, correlation function refers to the expectation value of the product of outcomes of multiple random variables. Suppose there are N observers, then the quantum correlation function can be expressed in the following equation:

$$\langle r_1 ... r_n \rangle = \mathrm{Tr}(\varrho \sigma_{j_i} \otimes ... \otimes \sigma_{j_n}) \tag{66}$$

where σ_{j_i} is the pauli opertator.

5.1 GME without multipartite correlation functions for odd number of qubits

Kaszlikowski et al [6] showed that for any odd number of qubits, one can generate a state that does not have n-partite correlations. One of the simpler example of this is the state $\varrho_N = \frac{1}{2}(|W\rangle\langle W| + |\bar{W}\rangle\langle\bar{W}|)$. For three qubits $|W\rangle$ is defined as $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $|\bar{W}\rangle$ is its 'antistate', generated from applying σ_x gate on each individual qubit. While Kaszlikowski et al [6] had stated that the state is genuinely multipartite entangled, we still tested the 3 qubit state using the criteria (40) and (41). (41) shows a positive result for genuinely multipartite entanglement while (41) failed in the detection. This is expected since (41) has a higher noise tolerance threshold. The results are shown below:

- For criterion (40),
  $|\varrho_{1,8}| \leq \sqrt{\varrho_{2,2}\varrho_{7,7}} + \sqrt{\varrho_{3,3}\varrho_{6,6}} + \sqrt{\varrho_{4,4}\varrho_{5,5}}$
  The L.H.S yields 0 while the R.H.S yields $\frac{1}{2}$.
- For criterion (41),
  $|\varrho_{2,3}|+|\varrho_{2,5}|+|\varrho_{3,5}| \leq \sqrt{\varrho_{1,1}\varrho_{4,4}}+\sqrt{\varrho_{1,1}\varrho_{6,6}}+\sqrt{\varrho_{1,1}\varrho_{7,7}}+\frac{1}{2}(\varrho_{2,2}+\varrho_{3,3}+\varrho_{5,5})$
  The L.H.S yields $\frac{1}{2}$ while the R.H.S yields $\frac{1}{4}$, resulting in the violation of the inequality.

The next step would be to show that for ϱ_N there is no N-fold correlation. We shall adopt the proof by Bennett et al [7] for our case. First, define T_μ such that,

$$T_\mu = \mathrm{Tr}\Big[\varrho(\sigma^x_{\mu_i} \otimes \sigma^x_{\mu_{i+1}} \otimes \sigma^z_{\mu_i} \otimes ... \otimes \sigma^z_{\mu_N})\Big] \tag{67}$$

where $\mu_i = \{0, 1...N\}$
μ_i in this case represents the number of the measurement.
Consider the following relations:

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |100...0\rangle = |010...0\rangle \tag{68}$$

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |010...0\rangle = |100...0\rangle \tag{69}$$

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |001...0\rangle = -|111...0\rangle \tag{70}$$

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |011...1\rangle = (-1)^N |101...1\rangle \tag{71}$$

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |101...1\rangle = (-1)^N |011...1\rangle \tag{72}$$

$$\sigma^x_1\sigma^x_2\sigma^z_3...\sigma^z_N |110...1\rangle = -(-1)^N |000...1\rangle \tag{73}$$

Hence,

$$\langle W|(\sigma^x_{\mu_i}\otimes\sigma^x_{\mu_{i+1}}\otimes\sigma^z_{\mu_i}\otimes...\otimes\sigma^z_{\mu_N})|W\rangle = \frac{2}{n} \tag{74}$$

$$\left\langle\bar{W}\right|(\sigma^x_{\mu_i}\otimes\sigma^x_{\mu_{i+1}}\otimes\sigma^z_{\mu_i}\otimes...\otimes\sigma^z_{\mu_N})\left|\bar{W}\right\rangle = (-1)^N\frac{2}{n} \tag{75}$$

At the same time,

$$T_\mu = \langle W|(\sigma^x_{\mu_i}\otimes\sigma^x_{\mu_{i+1}}\otimes\sigma^z_{\mu_i}\otimes...\otimes\sigma^z_{\mu_N})|W\rangle + \left\langle\bar{W}\right|(\sigma^x_{\mu_i}\otimes\sigma^x_{\mu_{i+1}}\otimes\sigma^z_{\mu_i}\otimes...\otimes\sigma^z_{\mu_N})\left|\bar{W}\right\rangle \tag{76}$$

Combining (74), (75) and (76),

$$T_\mu = 0 \tag{77}$$

5.2 GME without multipartite correlation for even number of qubits

The states for even N that are genuinely multipartite entangled and have no N-fold correlations are not known to exist. We shall propose one such example and show that it is indeed genuniely multipartite entangled with our set of criteria. Consider $\varrho = \frac{1}{4}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4|)$ where,

$$|\psi_1\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \tag{78}$$

$$|\psi_2\rangle = \frac{1}{2}(|0111\rangle - |1011\rangle + |1101\rangle - |1110\rangle \tag{79}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0110\rangle) \tag{80}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |1001\rangle) \tag{81}$$

$$\tag{82}$$

Using our set of criteria, ϱ is tested to violate the criterion (41), which proves that it is indeed geuninely multipartite entangled. Let S equals $R.H.S - L.H.S$, then if S is negative, the criterion is violated. In our case, $S = -0.3125$. $|\psi_3\rangle$ and $|\psi_4\rangle$ does violate the set of criteria as they are indeed biseparable.

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes(|011\rangle + |110\rangle)) \tag{83}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|1\rangle\otimes(|100\rangle + |001\rangle)) \tag{84}$$

However, it appears that $|\psi_2\rangle$ does not violate the criterias even though it is genuinely multipartite entangled. Exploring further, it appears that our criteria is not well suited to detect states with a mixture of states with positive and negative coefficients. For instance, when we used the criteria to further test another state which is known to be genuine multipartite entangled, $|\psi_1\rangle = \frac{1}{2}(|0001\rangle - |0010\rangle + |0100\rangle - |1000\rangle)$ and the criterion is not violated as well.

6 Maximum dimension of a bounded subspace

In the understanding of the properties of genuine multipartite entanglement, a interesting question pops up; What is the bound on the maximum dimension of the subspaces that spans the Hilbert space such that there are no product states in any permutation? In other words, what is the maximum dimension of subspaces with all state having a Schimdt rank of 2 in all permutations? There have been several studies done on a similar subject, which is the maximum bounded subspace such that there are no product states which is a less restricted problem. In this report, we shall give a brief review on one such studies.

6.1 Dimension of subspaces with bounded Schmidt rank

Winter et al [8], provided a proof based on parameter counting and supplemented it with a method to construct such a subspace for a bipartite system. Consider a bipartite system with dimension given by $d_A \times d_B$. Then the maximum dimension of the subspace such that each state has a Schmidt rank of r is given by $(d_A - r + 1)(d_B - r + 1)$. To understand the method of counting of the possible number of subspaces, it is essential to establish the mathematical framework and vector space to work in.

- Affine space:
  A affine space A^n over a field K is vector field K^n where the origin does not have any special role.

- Variety:
  Variety is defined as the set of solutions to a system of equations.

- Affine Variety: The common zero locus of a collection of polynomials such that the sets of polynomials is in the algebraic ring over the field K; $f \in k[x_1, x_1, x_2...x_n,]$.

  In a more approachable manner, we can use some examples.
  1. Consider a 2 dimensional affine space A^2 over the real field R. Polynomials in the ring $R[x, y]$ can then be viewed as functions on A^2 Suppose the set of polynomials has the elements $\{f_1(x, y), f_2(x, y)\}$ where, $f_1 = x^2 + y^2 - 1$ and $f_2 = 2x - y$, then sets of point $\{(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))\}$ on A^2 will be its variety. In essence, the variety is the interception points of the various curve or polynomials.

- Projective space:
  The projective space over a field K is the set of one-dimensional subspaces of the vector space K^{n+1}. In other words, the projective space P^n is the set of lines that pass through the origin of the space defined.

- Projective Varieties:
  The definition of projective variety is somewhat a huge chunk to digest. Formally, the projective variety over a algebraically closed field k is of a

projective space, P^n is the zero locus of a collection of homogeous polynomials F_d. Where any point in P^n is written as a homogeneous vector $[Z_0.....Z_n]$ and F is homogeneous of degree d such that,

$$F(\lambda Z_0....\lambda Z_n) = \lambda^d F(Z_0.....Z_n) \tag{85}$$

The rest of it is actually quite similiar to affine variety.

- Order-r-minor:
  This refers to the determinant of $r \times r$ submatrix.

- Determintant variety:
  The determinant variety over an algebraically closed field k in the space $k^{d_A d_B}$ is the zero locus of all order-r-minors of a given $d_A \times d_B$ matrix.

- Totally non-singular martix:
  A matrix is totally non-singular if all its minor are non-zero. One of its special properties is that in a $M \times M$ totally non singular matrix, any linear combination of n of the columns of M, denoted by ν contains at most $n-1$ zero elements, if $n < M$.

  One can prove this by contradiction. Suppose that the opposite is true, and there is more than $n-1$ zero elements in ν. Then one arranges the columns of the set of n columns while deleting the rows in M, whose row indices equate to the indices of non zero elements in ν. This will generate a submatrix that has linearly dependent columns, which contradicts the definition of a totally non-singular martrix.

- Vandermonde matrix: A vandermonder $n \times n$ matrix is one that is the form;

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & ... & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & ... & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & ... & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & ... & \alpha_n^{n-1} \end{bmatrix} \tag{86}$$

  One special defining structure of this type of matrix is that each element in the rows of the matrix follows a geometric progression. The matrix has another key property which is instrumental in our case; a Vandermonde matrix is totally non-singular iff the parameters $\alpha_1 ... \alpha_n$ are distinct. This arises from the determinant of A;

$$detA = \prod_{\substack{i,j=1 \\ i>j}}^{n} (\alpha_i - \alpha_j) \tag{87}$$

  While there are multiple proofs for (86), it will not be covered as it is not a main focus of this write up.

- Complex matrix:
  A complex matrix is one in which the matrix contains complex numbers. The complex numbers give rise to several interesting properties, especially in the underlying symmetries. It has been proven that an complex matrix is isomorphic to a Vandermonde matrix. This is done through Hadamards Maximum determinant. The full details are quite complicated and thus only a brief review will be done here.

  In the Hadamards Maximum determinant problem, one is interested in the largest possible determinant for any $n \times n$ square complex matrix whose element lies in a closed unit disk of 1. The bound is determined to be close to the following relationship by Hadamards in 1893;

  $$|det| \leq n^{\frac{n}{2}} \tag{88}$$

  This bound is isomorphic to the bound obtain by Vandermonde matrix of n roots of unity. As it is purely a mathematical interpretation, we will turn our focus to the physical significance of a totally non singular complex matrix. Suppose we have a state $|\Psi\rangle$ such that:

  $$|\Psi\rangle = a_1 |0000\rangle + a_2 |0001\rangle + ... + a_{16} |1111\rangle \tag{89}$$

  Writing a_n into a matrix M,

  $$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \tag{90}$$

The rationale behind the usage of projective space is that every state in the Hilbert space regardless of its coefficent can be represented by an element in the projective space. These elements can be put to a matrix and the solution to each sub-matrix or minor forms a variety in that space with these sub-matrices corresponding to subspaces in the Hilbert space. The rank of the given matrices can then be worked out using properties of determinant, which in turn gives the Schmidt rank of the subspace. Since the rank of the martix is isomorphic to the Schmidt rank, the bound on the Schmidt rank is isomorphic to the bound of the rank of the matrices. Using this property, one can identify the dimension of the determinant variety and thus determine the dimension of the bound on the Schmidt rank.

Proposition 1:
The variety of the set of $m \times n$ matrices of rank at most r is such that each minor has a rank of less then r is irreducible of codimension of $(m-r)(n-r)$.[9]

Since we are mainly interested in the determinant variety, we can first tweat proposition 1 into: The variety of the set of $m \times n$ matrices of rank at less than

r is such that each minor has a rank of less then r is irreducible of codimension of $(m-r-1)(n-r-1)$
As the determinant variety is defined by the set of vanishing order r minors, the size of this determinant variety will be defined as $mn-(m-r-1)(n-r-1)$. If the chosen subspace has a dimension of more then $(m-r-1)(n-r-1)$, then one can be sure that at least a state with Schmidt rank less then r will exist in that subspace, since $\mathbb{P}(S) \cap \mathbb{P}(D) < 0$. This is a result from the requirement that the $dim\mathbb{P}(S) + dim\mathbb{P}(D) \geq dim\mathbb{P}^{mn}$ and $dim\mathbb{P}(S) > (m-r-1)(n-r-1)-1$ by definition. $\mathbb{P}$ here represents the projective space.

The next step would be to show that one can always construct a bipartitie system, such that there is subspace with dimension of $(m-r-1)(n-r-1)$ or $(d_A-r-1)(d_B-r-1)$ in a $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ complex space. To illustrate the steps, consider a bipartition tripartite system $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^4$. There are 3 ways to arrange the partitions; $A|BC, B|CA$ and $C|AB$. It is enough for us to just consider a single partition$(A|BC)$ in our illustration as the steps are essentially the same. Consider the following quantum state:

$$|\Psi_3\rangle = a_1 |000\rangle + a_2 |001\rangle + ... + a_8 |111\rangle \tag{91}$$

Then one can put the coefficients into a matrix:

$$C_{A|BC} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \tag{92}$$

The next step would be to label the diagonals of $M_{A|BC}$ starting from the bottom left and write the diagonals in to a set of vectors,

$$\nu_{11} = \begin{bmatrix} a_5 \end{bmatrix} \tag{93}$$

$$\nu_{12} = \begin{bmatrix} a_1 \\ a_6 \end{bmatrix} \tag{94}$$

$$\nu_{13} = \begin{bmatrix} a_2 \\ a_7 \end{bmatrix} \tag{95}$$

$$\nu_{14} = \begin{bmatrix} a_3 \\ a_8 \end{bmatrix} \tag{96}$$

$$\nu_{15} = \begin{bmatrix} a_4 \end{bmatrix} \tag{97}$$

$$\tag{98}$$

If the length of the vector is larger or equal to r, construct a matrix with the vector in the position as its original diagonal position in $M_{A|BC}$. This will create a set of matrices with rank at least r that are linearly independent to each other, and in our example,

$$M_{11} = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 \end{bmatrix} \tag{99}$$

$$M_{12} = \begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & a_7 & 0 \end{bmatrix} \tag{100}$$

$$M_{13} = \begin{bmatrix} 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_8 \end{bmatrix} \tag{101}$$

$$\tag{102}$$

Thus we can create a new set of quantum states from this set of matrices that have no product states in the partition $A|BC$, namely,

$$|\psi_1\rangle = \alpha(a_1 |000\rangle + a_6 |101\rangle) \tag{103}$$

$$|\psi_2\rangle = \beta(a_2 |001\rangle + a_7 |110\rangle) \tag{104}$$

$$|\psi_3\rangle = \gamma(a_3 |010\rangle + a_8 |111\rangle) \tag{105}$$

$$|\Psi_{3'}\rangle = |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle \tag{106}$$

where α, β, γ are arbituary constants and $|\Psi_{3'}\rangle$ is the desired state with a subspace of dimension 3. This is a less formal and more illustrative construction as compared to the more generalised derivation in the Winter et al paper. Looking from another perspective, it is intuitive to see that one can obtain another set of matrices when take the anti-diagonals instead. The new set will consist of the matrix below,

$$M_{11'} = \begin{bmatrix} 0 & a_2 & 0 & 0 \\ a_5 & 0 & 0 & 0 \end{bmatrix} \tag{107}$$

$$M_{12'} = \begin{bmatrix} 0 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & 0 \end{bmatrix} \tag{108}$$

$$M_{13'} = \begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & a_7 & 0 \end{bmatrix} \tag{109}$$

$$\tag{110}$$

This observation does not add value into the generalised derivation of Winter et al if one is just considering the subspaces in a single partition, since the dimension of this subspace will not change whether one takes the diagonals or anti-diagonals in the construction. The significance of this difference will come in later in our construction later.

As we are aimming to construct a state that is genuine multi-partite entangled, we have to start considering other partition, namely $B|CA$ and $C|AB$ in our tripartite example. Using the same construction method for the other partitions with the tripartite system defined as in (83), we obtain 2 new coefficent matrices,

$$C_{B|CA} = \begin{bmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_7 & a_8 \end{bmatrix} \tag{111}$$

$$C_{C|AB} = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 \\ a_2 & a_4 & a_6 & a_8 \end{bmatrix} \tag{112}$$

If we compare the set of vector that has length more then r generated form the partition $B|CA$,

$$\nu_{22} = \begin{bmatrix} a_1 \\ a_4 \end{bmatrix} \tag{113}$$

$$\nu_{23} = \begin{bmatrix} a_2 \\ a_7 \end{bmatrix} \tag{114}$$

$$\nu_{24} = \begin{bmatrix} a_5 \\ a_8 \end{bmatrix} \tag{115}$$

with that of partition $C|AB$,

$$\nu_{32} = \begin{bmatrix} a_1 \\ a_4 \end{bmatrix} \tag{116}$$

$$\nu_{33} = \begin{bmatrix} a_3 \\ a_7 \end{bmatrix} \tag{117}$$

$$\nu_{34} = \begin{bmatrix} a_5 \\ a_8 \end{bmatrix} \tag{118}$$

as well as those from partition $A|BC$, there is no interception between the 3 sets of vectors. This is where the significance of the anti-diagonal construction comes in. With the anti-diagonal construction, we included all the combinatrics of 6 sets of vectors. As the construction from the anti-diagonals and diagonals of every partition is strictly non intercepting with one another, one can just ignored them. However the interception between set of vectors from diagonal of one partition with the set of vectors from the anti-diagonal of another parition is not a null set. However, even with this inclusion, it is still not possible to find any states that are common in the 6 sets. From another perspective, this method of construction provides us another simple witness if a state $|\psi_i\rangle$, is biseparable. By comparing the states with 3 matrices of coefficients, one can easily obtain the results, since if it is biseparable, the components will not have the same rolls or columns in all 3 matrices. The simplest case would be the GHZ state, with the coeffcients a_1 and a_8. Thus we can use this to eliminate the biseparate states, obtaining substates that are each geniune multipartite entangled. The total possible number of states will then reduce from 28 to 6. However, it is still not possible obtain the desired results.

6.2 Alternative Construction

From observation of the general construction of the tripartite states, we made an interesting observation and managed to create a GME state that indeed fit the required bound on the subspace for the tri-partite system. Thus we propose the following states,

$$|\psi_{1'}\rangle = \alpha'(|000\rangle + |111\rangle) \tag{119}$$
$$|\psi_{2'}\rangle = \beta'(|001\rangle + |010\rangle + |100\rangle) \tag{120}$$
$$|\psi_{3'}\rangle = \gamma'(|011\rangle + \omega\,|110\rangle + \omega^2\,|101\rangle) \tag{121}$$

where $\omega = e^{i\theta}, \theta \neq 2n\pi$ and α', β', γ' are arbitary constants.
We shall then prove that the state $|\psi_{GME'}\rangle = |\psi_{1'}\rangle + |\psi_{2'}\rangle + |\psi_{3'}\rangle$ is genuine multipartite entangled. Suppose $|\psi_{GME'}\rangle$ is separable for the bi-partition $A|BC$:

$$|\Psi\rangle = (a\,|0\rangle + b\,|1\rangle) \otimes (c_1\,|00\rangle + c_2\,|01\rangle + c_3\,|10\rangle + c_4\,|11\rangle) \tag{122}$$

By comparison, one obtain the following set of equations:

$$\alpha' = ac_1 = bc_4 \tag{123}$$
$$\beta' = ac_2 = ac_3 = bc_1 \tag{124}$$
$$\gamma' = ac_4 = \omega bc_3 = \omega^2 bc_2 \tag{125}$$

From equation (116), one observe that

$$c_2 = c_3 \tag{126}$$

However, from equation (117),

$$\omega c_2 = c_3 \tag{127}$$

Since ω is non zero, the only viable solution has $c_2 = 0$, which is a trival result, since if c_2 is 0, c_1, c_3, c_4 have to be 0, thus $|\Psi\rangle$= 0. Hence $|\Psi\rangle$ is non bi-separable in the $A|BC$ partition. The steps are repeated for the other 2 partitions. In the case of second partition $B|CA$, one obtains the following set of equations after comparision:

$$\alpha' = ac_1 = bc_4 \tag{128}$$
$$\beta' = ac_2 = ac_3 = bc_1 \tag{129}$$
$$\gamma' = \omega^2 ac_4 = \omega bc_3 = bc_2 \tag{130}$$

Again,

$$c_2 = c_3 \tag{131}$$

The difference comes from equation (122), such that now,

$$c_2 = \omega c_3 \tag{132}$$

which is similar to the case for $A|BC$ partition. This shows that for $B|CA$, $|\Psi\rangle$ is non bi-separable. Lastly, we have for the final partition we shall focus on just γ', as the rest are just repetition due to permutational symmetry.

$$\gamma' = \omega ac_4 = \omega^2 bc_3 = bc_2 \tag{133}$$

Hence,

$$c_2 = \omega^2 c_3 \tag{134}$$

Which is indeed the conclusion that was predicted. The violation is similar to $A|BC$ and $B|CA$ partition. Thus we can conclude that any state being a superposition of (119)-(121) is GME. Hence there are states span a 3-dimensional subspace with only GME states. The next key step is to understand how and why this works. This method is tested for 2 partite systems and it does not work for 2 partite system however. We hypothesised that this is due to the combination of both $|W\rangle$ and $|\bar{W}\rangle$ states allows one to form contradictions and for 2 partite systems, there is no $|W\rangle$ and $|\bar{W}\rangle$ states. While this appears to work for 4-partite system to obtain a subspace, the size of the subspace for the states created would then be much lesser then bound on the subspaces proposed by Winter et al which is 7. Moreover this method of construction does not provide a definite proof for bound on the subspace.

6.3 Possible future developments

The key to finding a mathematical and analytical solution to the bound on the subspaces lies on the bound on the dimensions of determinant variety. For Winter et al, they are only interested in just one partition of the system. Thus proposition 1 is not enough for the case of genuine multipartite entanglement, since for each different partition the matrices are different even though the size of the matrices and elements are the same. Fundamentally, one can imagine the solutions as a bundle of fibre, with each fibre crisscrossing at the various partitions and the determinant variety as the solution created when one take a slice at the bundle when the fibre crisscrossed and counts the number of points inside a certain 'area', which in our case, the 'area' refers to the Schmidt rank. Thus one has to find the maximum number of points that are enclosed in this 'area' no matter how the bundle crisscrosses. Plainly, one has to find the cardinal of the interception of the determinant varieties of the different partitions. One possible method could be the usage of Segre embedding which allows the mapping of varieties to a higher dimension geometry object.

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